MULTIVECTOR FIELD FORMULATION OF HAMILTONIAN FIELD THEORIES: EQUATIONS AND SYMMETRIES

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Abstract

We state the intrinsic form of the Hamiltonian equations of first-order Classical Field theories in three equivalent geometrical ways: using multivector fields, jet fields and connections. Thus, these equations are given in a form similar to that in which the Hamiltonian equations of mechanics are usually given.

Then, using multivector fields, we study several aspects of these equations, such as the existence and non-uniqueness of solutions, and the integrability problem. In particular, these problems are analyzed for the case of Hamiltonian systems defined in a submanifold of the multimomentum bundle. Furthermore, the existence of first integrals of these Hamiltonian equations is considered, and the relation between *Cartan-Noether symmetries* and *general symmetries* of the system is discussed. Noether's theorem is also stated in this context, both the "classical" version and its generalization to include higher-order Cartan-Noether symmetries. Finally, the equivalence between the Lagrangian and Hamiltonian formalisms is also discussed.

Key words: *Jet bundles, Multivector Fields, Connections, First order Field theories, Hamiltonian formalism, Symmetries, Noether's Theorem.*

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1 Introduction

The geometric structures underlying the covariant Lagrangian description of first-order Field theories are first order jet bundles $J^1E \stackrel{\pi^1}{\to} E \stackrel{\pi}{\to} M$ and their canonical structures (see [8], and references quoted therein). For the covariant Hamiltonian formalism several formulations arise, which use different kind of differentiable structures (*polisymplectic, k-symplectic, k-cosymplectic* or *multisymplectic* forms) and *multimomentum phase spaces* where the formalism takes place (see, for instance, [1], [5], [13], [15], [16], [23], [24], [28], [31], [34]).

In any case, a subject of interest in the geometrical description of the Hamiltonian formalism of Classical Field theories is related to the field equations, which are called the *Hamiltonian equations*. In the multisymplectic models, both in the Lagrangian and Hamiltonian formalisms, the field equations are usually written using the multisymplectic form in order to characterize the critical sections which are solutions of the problem [8], [12], [14]. This characterization can be derived from a suitable variational principle.

However, other attempts have been made to write these field equations in a more geometric-algebraic manner (as is done in mechanics, using vector fields); namely: by using *Ehresmann connections* [25], [36], *jet fields* [8], or *multivector fields* [14], [20], [21], [22], [23]. All of them have been carefully studied in [9] for the Lagrangian formalism of Field theories, and their equivalence demonstrated. The aim of this work is to carry out the analysis of these procedures for the Hamiltonian formalism, proving that all of them are equivalent, and using in particular the multivector field formulation to study the existence and non-unicity of solutions of these equations, and their integrability. Furthermore, equivalence theorems between the Lagrangian and Hamiltonian formalisms are stated. Thus, previous works of I.V. Kanatchikov devoted to the analysis of the field equations in the Hamiltonian formalism using multivector fields (in a more specific context), are completed.

Another subject of interest is the study of symmetries. Again using the multivector field formalism, we introduce and characterize different kinds of symmetries which are relevant in Field theory, showing their relation. In particular, Noether's theorem is proved and generalized in order to include higher-order Noether symmetries.

The paper is structured as follows: In Section 2 we review the construction of Hamiltonian systems in Field theory. Section 3 is devoted to setting the Hamiltonian field equations in terms of multivector fields, connections and jet fields (showing the equivalence of three methods), analyzing the existence and non-uniqueness of solutions (in the regular case), and their integrability. Sections 4 and 5 deal with the study of symmetries, first integrals and Noether's theorem. In Section 6, the case of restricted Hamiltonian systems is considered (those where the equations are defined in a submanifold of the multimomentum bundle). Hamiltonian systems associated with Lagrangian systems are treated in Section 7, including the equivalence between the Lagrangian and Hamiltonian formalism (for hyper-regular case). In Section 8, an example which is a quite general version of many typical models in Field theories is analyzed. The last Section is devoted to presenting the conclusions. The work ends with an appendix where the main features concerning multivector fields and connections are reviewed.

All manifolds are real, paracompact, connected and C^∞. All maps are C^∞. Sum over crossed repeated indices is understood. Throughout this paper $\pi\colon E \to M$ will be a fiber bundle ($\dim M = m$, $\dim E = N + m$), where M is an oriented manifold with volume form $\omega \in \Omega^m(M)$, and $\pi^1\colon J^1E \to E$ will be the jet bundle of local sections of π. The map $\bar{\pi}^1 = \pi \circ \pi^1\colon J^1E \longrightarrow M$ defines another structure of differentiable bundle. Finally, (x^μ, y^A, v^A_μ) will be natural local systems of coordinates in J^1E ($\mu = 1, \dots, m$; $A = 1, \dots, N$).

2 Hamiltonian systems

The Hamiltonian formalism for first-order Field theories requires the choice of a multimomentum phase space. This choice is not unique. In [10] and [11], the relations among some of them are shown, and in particular the following result is proved (see also [5] and [30]):

Theorem 1 *Let $\pi\colon E \to M$ be a fiber bundle. Then the following bundles are diffeomorphic:*

1. *$\Lambda_1^m \mathrm{T}^*E/\pi^*\Lambda^m\mathrm{T}^*M$ (where $\Lambda_1^m\mathrm{T}^*E \equiv \mathcal{M}\pi$ is the bundle of m-forms on E vanishing by the action of two π-vertical vector fields).*
2. *$\mathrm{Aff}(J^1E, \pi^*\Lambda^m\mathrm{T}^*M)/\pi^*\Lambda^m\mathrm{T}^*M$ (where $\mathrm{Aff}(J^1E, \pi^*\Lambda^m\mathrm{T}^*M)$ denotes the set of affine bundle maps from J^1E to $\pi^*\Lambda^m\mathrm{T}^*M$).*
3. *$\pi^*\mathrm{T}M \otimes \mathrm{V}^*(\pi) \otimes \pi^*\Lambda^m\mathrm{T}^*M$ (where $\mathrm{V}^*(\pi)$ denotes the dual bundle of $\mathrm{V}(\pi) = \ker\,\mathrm{T}\pi$).*

Thus, we take these equivalent bundles as our multimomentum phase space, and call it the *multimomentum bundle*. We denote it by $J^{1*}E$, and its points as $\tilde{y} \in J^{1*}E$. For the natural projections we will write $\tau^1\colon J^{1*}E \to E$ and $\bar{\tau}^1 = \pi \circ \tau^1\colon J^{1*}E \to M$. Given a system of coordinates adapted to the bundle $\pi\colon E \to M$, we can construct natural coordinates in $J^{1*}E$ and $\mathcal{M}\pi$, which will be denoted as (x^μ, y^A, p_A^μ) and (x^μ, y^A, p_A^μ, p), respectively.

In order to complete the geometric background of the Hamiltonian formalism, the multimomentum bundle must be endowed with a geometric structure which characterizes the system. Thus, we can construct *Hamiltonian systems* in three different ways [5], [11], [13], [26], [34]:

First, the multicotangent bundle $\Lambda^m\mathrm{T}^*E$ is endowed with canonical forms [4]: $\boldsymbol{\Theta} \in \mathit{\Omega}^m(\Lambda^m\mathrm{T}^*E)$ and the multisymplectic form $\boldsymbol{\Omega} := -\mathrm{d}\boldsymbol{\Theta} \in \mathit{\Omega}^{m+1}(\Lambda^m\mathrm{T}^*E)$. But $\mathcal{M}\pi \equiv \Lambda_1^m\mathrm{T}^*E$ is a subbundle of $\Lambda^m\mathrm{T}^*E$. Then, if $\lambda\colon \Lambda_1^m\mathrm{T}^*E \hookrightarrow \Lambda^m\mathrm{T}^*E$ is the natural imbedding, $\Theta := \lambda^*\boldsymbol{\Theta}$ and $\Omega := -\mathrm{d}\Theta = \lambda^*\boldsymbol{\Omega}$ are canonical forms in $\mathcal{M}\pi$, which are called the *multimomentum Liouville m and $(m+1)$ forms* of $\mathcal{M}\pi$. In a system of natural coordinates in $\mathcal{M}\pi$ we have

$$\Theta = p_A^\mu \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu + p\,\mathrm{d}^m x \quad , \quad \Omega = -\mathrm{d}p_A^\mu \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu - \mathrm{d}p \wedge \mathrm{d}^m x \tag{1}$$

A section $h\colon J^{1*}E \to \mathcal{M}\pi$ of the projection $\mu\colon \mathcal{M}\pi \to J^{1*}E$ is called a *Hamiltonian section*. The *Hamilton-Cartan m and $(m+1)$ forms* associated with the Hamiltonian section h are

$$\Theta_h = h^*\Theta \quad ; \quad \Omega_h = h^*\Omega = -\mathrm{d}\Theta_h$$

Using natural coordinates in $J^{1*}E$, a Hamiltonian section is locally specified by a *local Hamiltonian function* $H \in \mathrm{C}^\infty(U)$, $U \subset J^{1*}E$, such that $h(x^\mu, y^A, p_A^\mu) \equiv (x^\mu, y^A, p_A^\mu, p = -H(x^\gamma, y^B, p_B^\nu))$. Therefore, if $\bar{\tau}^{1*}\omega = \mathrm{d}^m x \equiv \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m$, the Hamilton-Cartan forms take the local expressions

$$\Theta_h = p_A^\mu \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu - H\mathrm{d}^m x \quad , \quad \Omega_h = -\mathrm{d}p_A^\mu \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu + \mathrm{d}H \wedge \mathrm{d}^m x \tag{2}$$

where $\mathrm{d}^{m-1}x_\mu \equiv i\left(\dfrac{\partial}{\partial x^\mu}\right)\mathrm{d}^m x$.

A variational problem can be posed for the system $(J^{1*}E, \Omega_h)$: the states of the field are the sections of $\bar{\tau}^1$ which are critical for the functional $\mathbf{H}\colon \Gamma_c(M, J^{1*}E) \to \mathbb{R}$ defined by $\mathbf{H}(\psi) := \int_M \psi^*\Theta_h$, for every $\psi \in \Gamma_c(M, J^{1*}E)$; where $\Gamma_c(M, J^{1*}E)$ is the set of compact supported sections of $\bar{\tau}^1$. As is known [8], [11], these critical sections are characterized by the condition $\psi^* i(X)\Omega_h = 0$,

for every $X \in \mathfrak{X}(J^{1*}E)$, which in natural coordinates in $J^{1*}E$, is equivalent to demanding that $\psi = (x^\mu, y^A(x), p^\mu_A(x))$ satisfies the equations

$$\frac{\partial y^A}{\partial x^\mu}\Big|_\psi = \frac{\partial H}{\partial p^\mu_A}\Big|_\psi \quad ; \quad \frac{\partial p^\mu_A}{\partial x^\mu}\Big|_\psi = -\frac{\partial H}{\partial y^A}\Big|_\psi \tag{3}$$

which are known as the *Hamilton-De Donder-Weyl equations*. But, as H is a local Hamiltonian function, these equations are not covariant; that is, they transform in a non-trivial way under changes of coordinates (see [5]).

The way to overcome this problem (and get a system of covariant equations) consists in using a connection. In fact, a connection ∇ in the bundle $\pi\colon E \to M$ induces a linear section $j_\nabla\colon J^{1*}E \to \mathcal{M}\pi$ of the projection μ [5], [11]. Then, we can construct the differentiable forms

$$\Theta^\nabla := j_\nabla^*\Theta \quad , \quad \Omega^\nabla := -\mathrm{d}\Theta^\nabla = j_\nabla^*\Omega$$

which are called the *Liouville* m and $(m+1)$ *forms* of $J^{1*}E$ associated with the connection ∇. Using natural coordinates in $J^{1*}E$ and $\mathcal{M}\pi$, if $\nabla = \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + \Gamma^A_\mu \frac{\partial}{\partial y^A}\right)$, then we have that $j_\nabla(x^\mu, y^A, p^A_\mu) = (x^\mu, y^A, p^A_\mu, p = -p^A_\nu \Gamma^\nu_A)$, and

$$\Theta^\nabla = p^\mu_A \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu - p^\mu_A \Gamma^A_\mu \mathrm{d}^m x \quad , \quad \Omega^\nabla = -\mathrm{d}p^\mu_A \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu + \mathrm{d}(p^\mu_A \Gamma^A_\mu) \wedge \mathrm{d}^m x$$

Now we have the following result:

Lemma 1 *If $h_1, h_2\colon J^{1*}E \to \mathcal{M}\pi$ are two sections of μ, then $h_1^*\Theta - h_2^*\Theta = h_1 - h_2$.*

(*Proof*) On the one hand, $h_1^*\Theta - h_2^*\Theta \in \varOmega^m(J^{1*}E)$. On the other hand, $h_1 - h_2\colon J^{1*}E \to \mathcal{M}\pi \equiv \Lambda^m_1 \mathrm{T}^*E$ has its image in $\pi^*\Lambda^m\mathrm{T}^*M$, because h_1, h_2 are sections of μ. But we have a natural inclusion $\pi^*\Lambda^m\mathrm{T}^*M \hookrightarrow \Lambda^m\mathrm{T}^*J^{1*}E$ given by means of the projection $\tau^1\colon J^{1*}E \to E$. Finally, the equality follows from a trivial calculation using natural coordinates. ■

Therefore, given a connection ∇ and a Hamiltonian section h, from this Lemma we have that

$$j_\nabla - h = j_\nabla^*\Theta - h^*\Theta \equiv \Theta^\nabla - \Theta_h := \mathcal{H}^\nabla_h$$

is a $\bar\tau^1$-semibasic m-form in $J^{1*}E$. It can be written as $\mathcal{H}^\nabla_h = \mathrm{H}\bar\tau^{1^*}\omega$, where $\mathrm{H} \in \mathrm{C}^\infty(J^{1*}E)$ is the *(global) Hamiltonian function* associated with $\mathcal{H}^\nabla_h$ and ω. Then, we can define

$$\Theta^\nabla_h := \Theta^\nabla - \mathcal{H}^\nabla_h \quad , \quad \Omega^\nabla_h := -\mathrm{d}\Theta^\nabla_{\mathcal{H}} = \Omega^\nabla + \mathrm{d}\mathcal{H}^\nabla_h$$

which are called the *Hamilton-Cartan* m and $(m+1)$ *forms* of $J^{1*}E$ associated with the Hamiltonian section h and the connection ∇. Their local expressions are

$$\begin{aligned}\Theta^\nabla_h &= p^\mu_A \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu - (\mathrm{H} + p^\mu_A \Gamma^A_\mu)\mathrm{d}^m x \\ \Omega^\nabla_h &= -\mathrm{d}p^\mu_A \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu + \mathrm{d}(\mathrm{H} + p^\mu_A \Gamma^A_\mu) \wedge \mathrm{d}^m x\end{aligned} \tag{4}$$

where H is a global Hamiltonian function, whose relation with the local Hamiltonian function H associated with the Hamiltonian section h is $\mathrm{H} = H - p^A_\mu \Gamma^\mu_A$ (in an open set U). In Field theory, every $\bar\tau^1$-semibasic m-form in $J^{1*}E$ is usually called a *Hamiltonian density*.

As in the above case, the variational problem for the system $(J^{1*}E, \Omega^\nabla_h)$ leads to the following characterization of the critical sections

which, in natural coordinates in $J^{1*}E$, is equivalent to the local equations (for the critical sections $\psi = (x^\mu, y^A(x), p^\mu_A(x))$)

$$\frac{\partial y^A}{\partial x^\mu}\Big|_\psi = \left(\frac{\partial \mathrm{H}}{\partial p^\mu_A} + \Gamma^A_\mu\right)\Big|_\psi \quad ; \quad \frac{\partial p^\mu_A}{\partial x^\mu}\Big|_\psi = -\left(\frac{\partial \mathrm{H}}{\partial y^A} + p^\nu_B \frac{\partial \Gamma^B_\nu}{\partial y^A}\right)\Big|_\psi \tag{6}$$

which are covariant, and are called the *Hamiltonian equations* of the system.

If, conversely, we take a connection ∇ and a Hamiltonian density $\mathcal{H}$, then making $j_\nabla - \mathcal{H} \equiv h_\nabla$ we obtain a section of μ, that is, a Hamiltonian section, because $\mathcal{H}\colon J^{1*}E \to \mathcal{M}\pi$ takes values in $\pi^*\Lambda^m\mathrm{T}^*M$. Hence we have proved the following:

Proposition 1 *A couple (h, ∇) in $J^{1*}E$ is equivalent to a couple $(\mathcal{H}, \nabla)$ (that is, given a connection ∇, Hamiltonian sections and Hamiltonian densities are in one-to-one correspondence).*

Bearing in mind this last result, we have a third way of obtaining a Hamiltonian system, which consists in giving a couple $(\mathcal{H}, \nabla)$, and then define

$$\Theta^\nabla_\mathcal{H} := \Theta^\nabla - \mathcal{H} \quad , \quad \Omega^\nabla_\mathcal{H} := -\mathrm{d}\Theta^\nabla_\mathcal{H} = \Omega^\nabla + \mathrm{d}\mathcal{H}$$

which are the *Hamilton-Cartan* m and $(m+1)$ *forms* of $J^{1*}E$ associated with the Hamiltonian density $\mathcal{H}$ and the connection ∇. Their local expressions are the same as in (4), with $\mathcal{H} = \mathrm{H}\bar{\tau}^{1*}\omega$.

Summarizing, there are three ways of constructing Hamiltonian systems in Field theory, namely:

- Giving a Hamiltonian section $h\colon J^{1*}E \to \mathcal{M}\pi$.
- Giving a couple (h, ∇), where h is a Hamiltonian section and ∇ a connection in $\pi\colon E \to M$.
- Giving a couple $(\mathcal{H}, \nabla)$, where $\mathcal{H}$ is a Hamiltonian density.

In each case, we can construct the Hamilton-Cartan forms and set a variational problem, which is called the *Hamilton-Jacobi principle* of the Hamiltonian formalism. As we have said, the second and third way are equivalent.

From now on, a couple $(J^{1*}E, \Omega^\nabla_h)$, or equivalently $(J^{1*}E, \Omega^\nabla_\mathcal{H})$, will be called a *Hamiltonian system*.

3 Hamiltonian equations, multivector fields and connections

We can set the Hamiltonian field equations using jet fields, connection forms and multivector fields (see the appendix A for notation and terminology).

First, an action of jet fields on forms is defined in the following way [8], [9]: consider the bundle $J^1(J^{1*}E)$ (the jet bundle of local sections of the projection $\bar{\tau}^1$), which is an affine bundle over $J^{1*}E$, whose associated vector bundle is $\bar{\tau}^{1*}\mathrm{T}^*M \otimes_E \mathrm{V}(\bar{\tau}^1)$. We have $J^1(J^{1*}E) \xrightarrow{\tau^1_1} J^{1*}E \xrightarrow{\bar{\tau}^1} M$. If $\mathcal{Y}\colon J^{1*}E \to J^1(J^{1*}E)$ is a jet field, a map $\bar{\mathcal{Y}}\colon \mathfrak{X}(M) \to \mathfrak{X}(J^{1*}E)$ can be defined as follows: for every $Z \in \mathfrak{X}(M)$, $\bar{\mathcal{Y}}(Z) \in \mathfrak{X}(J^{1*}E)$ is the vector field given by $\bar{\mathcal{Y}}(Z)(\tilde{y}) := (\mathrm{T}_{\bar{\tau}^1(\tilde{y})}\psi)(Z_{\bar{\tau}^1(\tilde{y})})$, for every $\tilde{y} \in J^{1*}E$ and $\psi \in \mathcal{Y}(\tilde{y})$. If $\mathcal{Y} \equiv (x^\mu, y^A, p^\mu_A, F^A_\mu(x,y,p), G^\rho_{A\mu}(x,y,p))$, its local expression is

$$\bar{\mathcal{Y}}\left(f^\mu \frac{\partial}{\ }\right) = f^\mu\left(\frac{\partial}{\ } + F^A \frac{\partial}{\ } + G^\rho \frac{\partial}{\ }\right)$$

This map induces an action of $\mathcal{Y}$ on the forms in $J^{1*}E$. In fact, let $\xi \in \Omega^{m+k}(J^{1*}E)$, with $k \geq 0$, we define $i(\mathcal{Y})\xi\colon \mathfrak{X}(M)\times \overset{(m)}{\dots} \times\mathfrak{X}(M) \longrightarrow \Omega^k(J^{1*}E)$ given by

$$[(i(\mathcal{Y})\xi)(Z_1,\dots,Z_m)](\tilde{y};X_1,\dots,X_k) := \xi(\tilde{y};\bar{\mathcal{Y}}(Z_1),\dots,\bar{\mathcal{Y}}(Z_m),X_1,\dots,X_k)$$

for $Z_1,\dots,Z_m \in \mathfrak{X}(M)$ and $X_1,\dots,X_k \in \mathfrak{X}(J^{1*}E)$. It is a $\mathrm{C}^\infty(M)$-linear and alternate map on the vector fields $Z_1,\dots,Z_m$. The $\mathrm{C}^\infty(J^{1*}E)$-linear map $i(\mathcal{Y})$ so defined, extended by zero to forms of degree $p<m$, is called the *inner contraction* with the jet field $\mathcal{Y}$. Then, it can be proved [8], [9] that:

Lemma 2 *If $\mathcal{Y}$ is an integrable jet field and $\xi \in \Omega^{m+1}(J^{1*}E)$. Then $i(\mathcal{Y})\xi = 0$ if, and only if, the integral sections $\psi\colon M \to J^{1*}E$ of $\mathcal{Y}$ satisfy the relation $\psi^*\, i(X)\xi = 0$, for every $X \in \mathfrak{X}(J^{1*}E)$.*

Theorem 2 *Let $(J^{1*}E,\Omega_h^\nabla)$ be a Hamiltonian system. The critical sections of the Hamilton-Jacobi principle are the sections $\psi \in \Gamma_c(M,J^{1*}E)$ satisfying any one of the following conditions:*

1. *They are the integral sections of an integrable jet field $\mathcal{Y}_\mathcal{H}\colon J^{1*}E \to J^1(J^{1*}E)$ satisfying that $i(\mathcal{Y}_\mathcal{H})\Omega_h^\nabla = 0$.*
2. *They are the integral sections of an integrable connection $\nabla_\mathcal{H}$ satisfying that $i(\nabla_\mathcal{H})\Omega_h^\nabla = (m-1)\Omega_h^\nabla$.*
3. *They are the integral sections of a class of integrable and $\bar{\tau}^1$-transverse multivector fields $\{X_\mathcal{H}\} \subset \mathfrak{X}^m(J^{1*}E)$ such that $i(X_\mathcal{H})\Omega_h^\nabla = 0$, for every $X_\mathcal{H} \in \{X_\mathcal{H}\}$.*

(*Proof*) Critical sections are characterized by the equation (5). Then, using the above lemma with $\xi \equiv \Omega_h^\nabla$, we obtain the equivalence between (5) and the item 1.

For the second item it suffices to use the expression in natural coordinates of a connection

$$\nabla_\mathcal{H} = \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + F_\mu^A\frac{\partial}{\partial y^A} + G_{A\mu}^\rho\frac{\partial}{\partial p_A^\rho}\right)$$

Hence, bearing in mind the local expression (4), we prove that the condition $i(\nabla_\mathcal{H})\Omega_h^\nabla = (m-1)\Omega_h^\nabla$ holds for an integrable connection if, and only if, the Hamiltonian equations (6) hold for its integral sections (see [25] and [36]).

Finally, item 3 is a direct consequence of the equivalence between orientable and integrable jet fields $\mathcal{Y}\colon J^{1*}E \to J^1(J^{1*}E)$, and classes of locally decomposable, $\bar{\tau}^1$-transverse and integrable multivector fields $\{X\} \subset \mathfrak{X}^m(J^{1*}E)$. ■

Thus, in Hamiltonian Field theories we search for (classes of) $\bar{\tau}^1$-transverse and locally decomposable multivector fields $X_\mathcal{H} \in \mathfrak{X}^m(J^{1*}E)$ such that:

1. The equation $i(X_\mathcal{H})\Omega_h^\nabla = 0$ holds.
2. $X_\mathcal{H}$ are integrable.

A representative of the class of multivector fields satisfying the first condition can be selected by demanding that $i(X_\mathcal{H})(\bar{\tau}^{1*}\omega) = 1$. Then its local expression is

$$X_\mathcal{H} = \bigwedge_{\mu=1}^{m}\left(\frac{\partial}{\partial x^\mu} + F_\mu^A\frac{\partial}{\partial y^A} + G_{A\mu}^\rho\frac{\partial}{\partial p_A^\rho}\right) \tag{7}$$

Concerning to the second condition, let us recall that, if $\{X_{\mathcal{H}}\} \subset \mathfrak{X}^m(J^{1*}E)$ is a class of locally decomposable and $\bar{\tau}^1$-transverse multivector fields, then $X_{\mathcal{H}}$ is integrable if, and only if, the curvature of the connection associated with this class vanishes everywhere.

Definition 1 *$X_{\mathcal{H}} \in \mathfrak{X}^m(J^{1*}E)$ will be called a* Hamilton-De Donder-Weyl (HDW) multivector field *for the system $(J^{1*}E, \Omega_h^{\nabla})$ if it is $\bar{\tau}^1$-transverse, locally decomposable and verifies the equation $i(X_{\mathcal{H}})\Omega_h^{\nabla} = 0$.*

We denote the set of HDW-multivector fields by $\mathfrak{X}^m_{HDW}(J^{1*}E, \Omega_h^{\nabla})$.

Theorem 3 (Existence and local multiplicity of HDW-multivector fields): *Let $(J^{1*}E, \Omega_h^{\nabla})$ be a Hamiltonian system.*

1. *There exist classes of HDW-multivector fields $\{X_{\mathcal{H}}\} \subset \mathfrak{X}^m_{HDW}(J^{1*}E)$, (and hence equivalent jet fields $\mathcal{Y}_{\mathcal{H}}\colon J^1E \to J^1(J^{1*}E)$ with associated connection forms $\nabla_{\mathcal{H}}$, satisfying that $i(\mathcal{Y}_{\mathcal{H}})\Omega_h^{\nabla} = 0$ and $i(\nabla_{\mathcal{H}})\Omega_h^{\nabla} = (m-1)\Omega_h^{\nabla}$, respectively).*
2. *In a local system the above solutions depend on $N(m^2-1)$ arbitrary functions.*

(*Proof*)

1. First we analyze the local existence of solutions and then their global extension.

   In a chart of natural coordinates in $J^{1*}E$, the expression of Ω_h^{∇} is (4); and taking the multivector field given in (7) as representative of the class $\{X_{\mathcal{H}}\}$, from the relation $i(X_{\mathcal{H}})\Omega_h^{\nabla} = 0$ we obtain the following conditions:

   - The coefficients on $\mathrm{d}p_A^{\mu}$ must vanish:
   $$0 = F_{\nu}^A - \frac{\partial \mathrm{H}}{\partial p_A^{\nu}} - \Gamma_{\nu}^A \qquad (\text{for every } A, \nu) \tag{8}$$
   This system of Nm linear equations determines univocally the functions F_{ν}^A.
   - The coefficients on $\mathrm{d}y^A$ must vanish
   $$0 = G_{A\mu}^{\mu} + \frac{\partial \mathrm{H}}{\partial y^A} + p_B^{\nu}\frac{\partial \Gamma_{\nu}^B}{\partial y^A} \qquad (A = 1, \ldots, N) \tag{9}$$
   which is a compatible system of N linear equations on the Nm^2 functions $G_{A\nu}^{\mu}$.
   - Using these results we obtain that the coefficients on $\mathrm{d}x^{\mu}$ vanish identically.

   These results allow us to assure the local existence of (classes of) multivector fields satisfying the desired conditions. The corresponding global solutions are then obtained using a partition of unity subordinated to a cover of $J^{1*}E$ made of natural charts.

   (Note that, if $\psi = (x^{\mu}, y^A(x^{\nu}), p_A^{\mu}(x^{\nu}))$ is an integral section of $X_{\mathcal{H}}$ (resp. $\mathcal{Y}_{\mathcal{H}}$), then
   $$F_{\mu}^A \circ \psi = \frac{\partial y^A}{\partial x_{\mu}} \quad ; \quad G_{A\mu}^{\mu} \circ \psi = -\frac{\partial p_A^{\mu}}{\partial x^{\mu}}$$
   and then equations (8) and (9) are the Hamiltonian equations for ψ.)

2. In natural coordinates in $J^{1*}E$, a HDW-multivector field $X_{\mathcal{H}} \in \{X_{\mathcal{H}}\}$ is given by (7). So, it is determined by the Nm coefficients F_ν^A (which are obtained as the solution of (8)), and by the Nm^2 coefficients $G_{A\nu}^\mu$, which are related by the N independent equations (9). Therefore, there are $N(m^2-1)$ arbitrary functions.

■

Finally we try to determine if it is possible to find a class of integrable HDW-multivector fields. Hence we must impose that the corresponding multivector field $X_{\mathcal{H}}$ verify the integrability condition; that is, the curvature of the associated connection $\nabla_{\mathcal{H}}$ vanishes everywhere, that is,

$$\begin{aligned} 0 \;=\; & \left(\frac{\partial F_\eta^B}{\partial x^\mu} + F_\mu^A \frac{\partial F_\eta^B}{\partial y^A} + G_{B\mu}^\gamma \frac{\partial F_\eta^B}{\partial p_A^\gamma} - \frac{\partial F_\mu^B}{\partial x^\eta} - F_\eta^A \frac{\partial F_\mu^B}{\partial y^A} - G_{A\eta}^\rho \frac{\partial F_\mu^B}{\partial p_A^\rho}\right) (\mathrm{d}x^\mu \wedge \mathrm{d}x^\eta) \otimes \frac{\partial}{\partial y^B} + \\ & \left(\frac{\partial G_{B\eta}^\rho}{\partial x^\mu} + F_\mu^A \frac{\partial G_{B\eta}^\rho}{\partial y^A} + G_{A\mu}^\gamma \frac{\partial G_{B\eta}^\rho}{\partial p_A^\gamma} - \frac{\partial G_{B\mu}^\rho}{\partial x^\eta} - F_\eta^A \frac{\partial G_{B\mu}^\rho}{\partial y^A} - G_{A\eta}^\gamma \frac{\partial G_{B\mu}^\rho}{\partial p_A^\gamma}\right) (\mathrm{d}x^\mu \wedge \mathrm{d}x^\eta) \otimes \frac{\partial}{\partial p_B^\rho} \end{aligned}$$

or, what is equivalent, the following system of equations hold (for $1 \leq \mu < \eta \leq m$)

$$\begin{aligned} 0 &= \frac{\partial F_\eta^B}{\partial x^\mu} + F_\mu^A \frac{\partial F_\eta^B}{\partial y^A} + G_{A\mu}^\gamma \frac{\partial F_\eta^B}{\partial p_A^\gamma} - \frac{\partial F_\mu^B}{\partial x^\eta} - F_\eta^A \frac{\partial F_\mu^B}{\partial y^A} - G_{A\eta}^\rho \frac{\partial F_\mu^B}{\partial p_A^\rho} \\ &= \frac{\partial^2 \tilde{\mathrm{H}}}{\partial x^\mu \partial p_B^\eta} + \frac{\partial H}{\partial p_A^\mu} \frac{\partial^2 H}{\partial y^A \partial p_B^\eta} + \\ & \quad G_{A\mu}^\gamma \frac{\partial^2 H}{\partial p_A^\gamma \partial p_B^\eta} - \frac{\partial^2 H}{\partial x^\eta \partial p_B^\eta} - \frac{\partial H}{\partial p_A^\eta} \frac{\partial^2 H}{\partial y^A \partial p_B^\mu} - G_{A\eta}^\rho \frac{\partial^2 H}{\partial p_A^\rho \partial p_B^\mu} \end{aligned} \tag{10}$$

$$\begin{aligned} 0 &= \frac{\partial G_{B\eta}^\rho}{\partial x^\mu} + F_\mu^A \frac{\partial G_{B\eta}^\rho}{\partial y^A} + G_{A\mu}^\gamma \frac{\partial G_{B\eta}^\rho}{\partial p_A^\gamma} - \frac{\partial G_{B\mu}^\rho}{\partial x^\eta} - F_\eta^A \frac{\partial G_{B\mu}^\rho}{\partial y^A} - G_{A\eta}^\gamma \frac{\partial G_{B\mu}^\rho}{\partial p_A^\gamma} \\ &= \frac{\partial G_{B\eta}^\rho}{\partial x^\mu} + \frac{\partial H}{\partial p_A^\mu} \frac{\partial G_{B\eta}^\rho}{\partial y^A} + G_{A\mu}^\gamma \frac{\partial G_{B\eta}^\rho}{\partial p_A^\gamma} - \frac{\partial G_{B\mu}^\rho}{\partial x^\eta} - \frac{\partial H}{\partial p_A^\eta} \frac{\partial G_{B\mu}^\rho}{\partial y^A} - G_{A\eta}^\gamma \frac{\partial G_{B\mu}^\rho}{\partial p_A^\gamma} \end{aligned} \tag{11}$$

(where $H \equiv \mathrm{H} + p_\mu^A \Gamma_A^\mu$, and use is made of the Hamiltonian equations). Since these additional conditions on the functions $G_{A\nu}^\mu$ must be imposed in order to assure that $X_{\mathcal{H}}$ is integrable, the number of arbitrary functions will be in general less than $N(m^2-1)$.

As far as we know, since this is a system of partial differential equations with linear restrictions, there is no way of assuring the existence of an integrable solution, or of selecting it. Observe that, considering the Hamiltonian equations for the coefficients $G_{A\nu}^\mu$(equations (9)), together with the integrability conditions (10) and (11), we have $N+\frac{1}{2}Nm(m-1)$ linear equations and $\frac{1}{2}Nm^2(m-1)$ partial differential equations. Then, if the set of linear restrictions (9) and (10) allow us to isolate $N+\frac{1}{2}Nm(m-1)$ coefficients $G_{A\nu}^\mu$ as functions on the remaining ones; and the set of $\frac{1}{2}Nm^2(m-1)$ partial differential equations (11) on these remaining coefficients satisfies the conditions on *Cauchy-Kowalewska's theorem* [6], then the existence of integrable HDW-multivector fields (in $J^{1*}E$)) is assured. If this is not the case, we can eventually select some particular HDW-multivector field solution, and apply the integrability algorithm developed in [9] in order to find a submanifold $\mathcal{I} \hookrightarrow J^{1*}E$ (if it exists), where this multivector field is integrable (and tangent to $\mathcal{I}$).

Other results concerning the expression of the Hamiltonian equations in terms of multivector fields can be found in [20], [21], [22] and [23], where the definition of Poisson algebras in Field theories is also given (see also [5]).

4 Symmetries and first integrals

Next we recover the idea of *first integral* or *conserved quantity*, and state Noether's theorem for Hamiltonian systems in Field theory, in terms of multivector fields. In this sense, a great part of our discussion is a generalization of the results obtained for non-autonomous (non-regular) mechanical systems (see, in particular, [27], and references quoted therein). We refer to appendix A to review the definition of the basic differential operations on the set of multivector fields in a manifold.

Consider a Hamiltonian system $(J^{1*}E, \Omega_h^\nabla)$. Let

$$\ker^m \Omega_h^\nabla := \{\mathcal{Z} \in \mathfrak{X}^m(J^{1*}E) \; ; \; i(\mathcal{Z})\Omega_h^\nabla = 0\}$$

and let $\ker^m_\omega \Omega_h^\nabla \subset \mathfrak{X}^m(J^{1*}E)$ be the set of m-multivector fields satisfying that

$$i(X)\Omega_h^\nabla = 0 \quad , \quad i(X)(\bar{\tau}^{1*}\omega) \neq 0 \tag{12}$$

These are $\bar{\tau}^1$-transverse multivector fields (but not locally decomposable, necessarily), and as usual we can select a representative on each equivalence class of solutions, by demanding that $i(X)(\bar{\tau}^{1*}\omega) = 1$. Remember that HDW-multivector fields are solutions of (12) which are locally decomposable. Then, if $\mathfrak{X}^m_{IHDW}(J^{1*}E, \Omega_h^\nabla)$ denotes the set of integrable HDW-multivector fields, we obviously have that

$$\mathfrak{X}^m_{IHDW}(J^{1*}E, \Omega_h^\nabla) \subset \mathfrak{X}^m_{HDW}(J^{1*}E, \Omega_h^\nabla) \subset \ker^m_\omega \Omega_h^\nabla \subset \ker^m \Omega_h^\nabla$$

Now we introduce the following terminology [12], [27]:

Definition 2 *A* first integral *or a* conserved quantity *of a Hamiltonian system $(J^{1*}E, \Omega_h^\nabla)$ is a form $\xi \in \Omega^{m-1}(J^{1*}E)$ such that $\mathrm{L}(X)\xi = 0$, for every $X \in \ker^m_\omega \Omega_h^\nabla$.*

Observe that, in this case, $\mathrm{L}(X)\xi = (-1)^{m+1}\, i(X)\mathrm{d}\xi$.

Proposition 2 *If $\xi \in \Omega^{m-1}(J^{1*}E)$ is a first integral of a Hamiltonian system $(J^{1*}E, \Omega_h^\nabla)$, and $X \in \ker^m_\omega \Omega_h^\nabla$ is integrable, then ξ is closed on the integral submanifolds of X. That is, if $j_S \colon S \hookrightarrow J^{1*}E$ is an integral submanifold of X, then $\mathrm{d}j_S^*\xi = 0$.*

(*Proof*) Let $X_1, \ldots, X_m \in \mathfrak{X}(J^{1*}E)$ be independent vector fields tangent to the (m-dimensional) integral submanifold S. Then $X = fX_1 \wedge \ldots \wedge X_m$, for some $f \in \mathrm{C}^\infty(J^{1*}E)$. Therefore, as $i(X)\mathrm{d}\xi = 0$, we have that

$$j_S^*[\mathrm{d}\xi(X_1, \ldots, X_m)] = j_S^*\, i(X_1 \wedge \ldots \wedge X_m)\mathrm{d}\xi = 0$$

■

Conserved quantities can be characterized as follows:

Proposition 3 *If $\xi \in \Omega^{m-1}(J^{1*}E)$ is a first integral of a Hamiltonian system $(J^{1*}E, \Omega_h^\nabla)$, then $\mathrm{L}(\mathcal{Z})\xi = 0$, for every $\mathcal{Z} \in \ker^m \Omega_h^\nabla$.*

(*Proof*) Consider the conditions (12), with $i(X)(\bar{\tau}^{1*}\omega) = 1$, and let $X_0 \in \ker^m_\omega \Omega_h^\nabla$ be a particular solution. Then, any other solution can be obtained by making $fX_0 + Z$, with $Z \in \ker^m \Omega_h^\nabla \cap \ker^m(\bar{\tau}^{1*}\omega)$ and $f \in \mathrm{C}^\infty(J^{1*}E)$. Thus we have that

Then, for every $Z \in \ker^m \Omega_h^\nabla \cap \ker^m (\bar{\tau}^{1*}\omega)$, we have that $Z = X_1 - X_2$, with $X_1, X_2 \in \ker^m_\omega \Omega_h^\nabla$ such that $i(X_1)(\bar{\tau}^{1*}\omega) = i(X_2)(\bar{\tau}^{1*}\omega)$. Hence, if ξ is a first integral, we have that $\mathrm{L}(Z)\xi = 0$. On the other hand, taking $X_0 \in \ker^m_\omega \Omega_h^\nabla$, for every $\mathcal{Z} \in \ker^m \Omega_h^\nabla$ we can write the identity

$$\mathcal{Z} = (\mathcal{Z} - i(\mathcal{Z})(\bar{\tau}^{1*}\omega)X_0) + i(\mathcal{Z})(\bar{\tau}^{1*}\omega)X_0$$

then, if $i(X_0)(\bar{\tau}^{1*}\omega) = 1$, it follows that $\mathcal{Z} - i(\mathcal{Z})(\bar{\tau}^{1*}\omega)X_0 \in \ker^m \Omega_h^\nabla \cap \ker^m (\bar{\tau}^{1*}\omega)$, hence

$$\mathrm{L}(\mathcal{Z})\xi = \mathrm{L}(\mathcal{Z} - i(\mathcal{Z})(\bar{\tau}^{1*}\omega)X_0)\xi + \mathrm{L}(i(\mathcal{Z})(\bar{\tau}^{1*}\omega)X_0)\xi = (-1)^{m+1}\, i(\mathcal{Z})(\bar{\tau}^{1*}\omega)\, i(X_0)\mathrm{d}\xi = 0$$

since $\mathrm{d}\, i(X_{\mathcal{H}})\xi_Y = 0$, because $\xi_Y \in \mathit{\Omega}^{m-1}(J^{1*}E)$. ∎

The converse of this statement holds obviously, and hence this is a characterization of first integrals.

Next we introduce the following terminology (which will be justified in Theorem 4):

Definition 3 *An* (infinitesimal) general symmetry *of a Hamiltonian system* $(J^{1*}E, \Omega_h^\nabla)$ *is a vector field* $Y \in \mathfrak{X}(J^{1*}E)$ *satisfying that* $[Y, \ker^m \Omega_h^\nabla] \subset \ker^m \Omega_h^\nabla$.

Bearing in mind the properties of multivector fields (see the Appendix), we obtain that general symmetries have the following basic properties:

- If $Y \in \mathfrak{X}(J^{1*}E)$ is a general symmetry, then so is $Y + Z$, for every $Z \in \ker \Omega_h^\nabla$.
- If $Y_1, Y_2 \in \mathfrak{X}(J^{1*}E)$ are general symmetries, then so is $[Y_1, Y_2]$.

A first characterization of general symmetries is given by:

Lemma 3 *Let* $(J^{1*}E, \Omega_h^\nabla)$ *be a Hamiltonian system,* $Y \in \mathfrak{X}(J^{1*}E)$*, and let* F_t *be a local flow of* Y*.* Y *is a general symmetry if, and only if,* $F_{t*}(\ker^m \Omega_h^\nabla) \subset \ker^m \Omega_h^\nabla$*, in the corresponding open sets.*

(*Proof*) As $\ker^m \Omega_h^\nabla$ is locally finite-generated, we can take a local basis $Z_1, \ldots, Z_r$ of $\ker^m \Omega_h^\nabla$, and then the assertion is equivalent to proving that $[Y, Z_i] = f_i^j Z_j$ if, and only if, $F_{t*}Z_i = g_i^j Z_j$ (for every $i = 1, \ldots, r$), where g_i^j are differentiable functions on the corresponding open set, also depending on t.

It is clear that, if $F_{t*}Z_i = g_i^j Z_j$, then $[Y, Z_i] = f_i^j Z_j$.

For the converse, we have to prove the existence of functions g_i^j such that $F_{t*}Z_i = g_i^j Z_j$. Suppose that $[Y, Z_i] = f_i^j Z_j$, and remember that $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} F_{t*}Z_i = F_{s*}[Y, Z_i]$. Hence, on the one hand we obtain

$$F_{s*}[Y, Z_i] = F_{s*}(f_i^j Z_j) = (F_s^{-1})^* f_i^j F_{s*}Z_j = (F_s^{-1})^* f_i^j (g_j^k Z_k)$$

and on the other hand, we have that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} F_{t*}Z_i = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} g_i^k Z_k = \left.\frac{\mathrm{d}g_i^k}{\mathrm{d}t}\right|_{t=s} Z_k$$

therefore, comparing these expressions, we conclude that

This is a system of ordinary linear differential equations for the functions g_i^k, which, with the initial condition $g_i^k(0) = \delta_i^k$, has a unique solution, defined for every t on the domain of F_t. Then, taking this solution, the result holds. ■

Using this Lemma, we can prove that:

Theorem 4 *Let $Y \in \mathfrak{X}(J^{1*}E)$ be a general symmetry of a Hamiltonian system $(J^{1*}E, \Omega_h^\nabla)$, and F_t a local flow of Y.*

1. *If $\mathcal{Z} \in \ker^m \Omega_h^\nabla$ is an integrable multivector field, then F_t transforms integral submanifolds of $\mathcal{Z}$ into integral submanifolds of $F_{t*}\mathcal{Z}$.*
2. *In particular, if $Y \in \mathfrak{X}(J^{1*}E)$ is $\bar{\tau}^1$-projectable, and $X_{\mathcal{H}} \in \mathfrak{X}^m_{IHDW}(J^{1*}E, \Omega_h^\nabla)$, then F_t transforms critical sections of $X_{\mathcal{H}}$ into critical sections of $F_{t*}X_{\mathcal{H}}$, and hence $F_{t*}X_{\mathcal{H}} \in \mathfrak{X}^m_{IHDW}(J^{1*}E, \Omega_h^\nabla)$.*

(*Proof*)

1. Let $X_1, \ldots, X_m \in \mathfrak{X}(J^{1*}E)$ be vector fields locally expanding the involutive distribution associated with $\mathcal{Z}$. Then $F_{t*}X_1, \ldots, F_{t*}X_m$ generate another distribution which is also involutive, and, hence, is associated with a class of locally decomposable multivector fields whose representative is just $F_{t*}\mathcal{Z}$, by construction. The assertion about the integral submanifolds is then immediate.
2. First observe that, as Y is $\bar{\tau}^1$-projectable, then F_t restricts to a local flow F_t^M in M; that is, we have $F_t^M \circ \bar{\tau}^1 = \bar{\tau}^1 \circ F_t$. Now, for every $\psi\colon M \to J^{1*}E$, integral section of $X_{\mathcal{H}}$, we can define $\psi_t\colon M \to J^{1*}E$ by the relation $F_t \circ \psi = \psi_t \circ F_t^M$, which is also a section of $\bar{\tau}^1$, because
$$\bar{\tau}^1 \circ \psi_t = \bar{\tau}^1 \circ F_t \circ \psi \circ (F_t^M)^{-1} = F_t^M \circ \bar{\tau}^1 \circ \psi \circ (F_t^M)^{-1} = F_t^M \circ (F_t^M)^{-1} = \mathrm{Id}_M$$
since $\bar{\tau}^1 \circ \psi = \mathrm{Id}_M$. Then, observe that, by construction, $\mathrm{Im}\,\psi_t = F_t(\mathrm{Im}\,\psi)$ is an integral submanifold of $F_{t*}X_{\mathcal{H}}$, and as is a section of $\bar{\tau}^1$, it is $\bar{\tau}^1$-transverse. Hence $F_{t*}X_{\mathcal{H}}$ (which belongs to $\ker^m \Omega_h^\nabla$, by Lemma 3) is integrable (then locally decomposable), and as its integral submanifolds are sections of $\bar{\tau}^1$, it is $\bar{\tau}^1$-transverse, thus $F_{t*}X_{\mathcal{H}} \in \mathfrak{X}^m_{IHDW}(J^{1*}E, \Omega_h^\nabla)$.

■

General symmetries can be used for obtaining conserved quantities, as follows:

Proposition 4 *If $\xi \in \Omega^{m-1}(J^{1*}E)$ is a first integral of a Hamiltonian system $(J^{1*}E, \Omega_h^\nabla)$, then so is $\mathrm{L}(Y)\xi$, for every general symmetry $Y \in \mathfrak{X}(J^{1*}E)$.*

(*Proof*) For every first integral $\xi \in \Omega^{m-1}(J^{1*}E)$, and $\mathcal{Z} \in \ker^m \Omega_h^\nabla$, if $Y \in \mathfrak{X}(J^{1*}E)$ is a general symmetry, we have that
$$\mathrm{L}(\mathcal{Z})\,\mathrm{L}(Y)\xi = \mathrm{L}([\mathcal{Z}, Y])\xi + \mathrm{L}(Y)\,\mathrm{L}(\mathcal{Z})\xi = \mathrm{L}([\mathcal{Z}, Y])\xi = 0$$
since $[\mathcal{Z}, Y] \in \ker^m \Omega_h^\nabla$, and as a consequence of Proposition 3. ■

5 Noether's theorem for multivector fields

There is another kind of symmetries which play a relevant role, as generators of conserved quantities:

Definition 4 *An* (infinitesimal) Cartan *or* Noether symmetry *of a Hamiltonian system* $(J^{1*}E, \Omega_h^\nabla)$ *is a vector field* $Y \in \mathfrak{X}(J^{1*}E)$ *satisfying that* $\mathrm{L}(Y)\Omega_h^\nabla = 0$.

Remarks:

- It is immediate to prove that, if $Y_1, Y_2 \in \mathfrak{X}(J^{1*}E)$ are Cartan-Noether symmetries, then so is $[Y_1, Y_2]$.
- Observe that the condition $\mathrm{L}(Y)\Omega_h^\nabla = 0$ is equivalent to demanding that $i(Y)\Omega_h^\nabla$ is a closed m-form in $J^{1*}E$. Therefore, for every $p \in J^{1*}E$, there exists an open neighborhood $U_p \ni p$, and $\xi_Y \in \Omega^{m-1}(U_p)$, such that $i(Y)\Omega_h^\nabla = \mathrm{d}\xi_Y$ (on U_p). Thus, a Cartan-Noether symmetry of a Hamiltonian system is just a *locally Hamiltonian vector field* for the multisymplectic form Ω_h^∇, and ξ_Y is the corresponding *local Hamiltonian form*, which is unique, up to a closed $(m-1)$-form.

Cartan-Noether symmetries have the following property:

Proposition 5 *Let* $Y \in \mathfrak{X}(J^{1*}E)$ *be a Cartan-Noether symmetry of a Hamiltonian system* $(J^{1*}E, \Omega_h^\nabla)$. *Therefore:*

1. $\mathrm{L}(Y)\Theta_h^\nabla$ *is a closed form, hence, in an open set* $U \subset J^{1*}E$, *there exist* $\zeta_Y \in \Omega^{m-1}(U)$ *such that* $\mathrm{L}(Y)\Theta_h^\nabla = \mathrm{d}\zeta_Y$.
2. *If* $i(Y)\Omega_h^\nabla = \mathrm{d}\xi_Y$, *in an open set* $U \subset J^{1*}E$, *then*

$$\mathrm{L}(Y)\Theta_h^\nabla = \mathrm{d}(i(Y)\Theta_h^\nabla - \xi_Y) = \mathrm{d}\zeta_Y \quad (\text{in } U)$$

(*Proof*)

1. The first item is immediate since $\mathrm{d}\,\mathrm{L}(Y)\Theta_h^\nabla = \mathrm{L}(Y)\mathrm{d}\Theta_h^\nabla = 0$.
2. For the second item we have

$$\mathrm{L}(Y)\Theta_h^\nabla = \mathrm{d}\,i(Y)\Theta_h^\nabla + i(Y)\mathrm{d}\Theta_h^\nabla = \mathrm{d}\,i(Y)\Theta_h^\nabla - i(Y)\Omega_h^\nabla = \mathrm{d}(i(Y)\Theta_h^\nabla - \xi_Y)$$

Hence we can write $\xi_Y = i(Y)\Theta_h^\nabla - \zeta_Y$ (up to a closed $(m-1)$-form).

■

Remark:

- As a particular case, if for a Cartan-Noether symmetry Y the condition $\mathrm{L}(Y)\Theta_h^\nabla = 0$ holds, we can take $\xi_Y = i(Y)\Theta_h^\nabla$. In this case Y is said to be an *exact Cartan-Noether symmetry*.

Proposition 6 *Every Cartan-Noether symmetry of a Hamiltonian system* $(J^{1*}E, \Omega_h^\nabla)$ *is a general symmetry.*

(*Proof*) Let $Y \in \mathfrak{X}(J^{1*}E)$ be a Cartan-Noether symmetry. For every $\mathcal{Z} \in \ker^m \Omega_h^\nabla$, we have that

$$i([Y, \mathcal{Z}])\Omega_h^\nabla = \mathrm{L}(Y)\, i(\mathcal{Z})\Omega_h^\nabla + (-1)^{2+m}\, i(\mathcal{Z})\, \mathrm{L}(Y)\Omega_h^\nabla = 0$$

and therefore $[Y, \mathcal{Z}] \subset \ker^m \Omega_h^\nabla$. ∎

Finally, the classical *Noether's theorem* of Hamiltonian mechanics can be generalized to Field theory as follows:

Theorem 5 (Noether): *If* $Y \in \mathfrak{X}(J^{1*}E)$ *is a Cartan-Noether symmetry of a Hamiltonian system* $(J^{1*}E, \Omega_h^\nabla)$*, with* $i(Y)\Omega_h^\nabla = \mathrm{d}\xi_Y$*. Then, for every HDW-multivector field* $X_\mathcal{H} \in \mathfrak{X}^m(J^{1*}E)$*, we have that*

$$\mathrm{L}(X_\mathcal{H})\xi_Y = 0$$

that is, any Hamiltonian $(m-1)$*-form* ξ_Y *associated with* Y *is a first integral of* $(J^{1*}E, \Omega_h^\nabla)$*.*

(*Proof*) If $Y \in \mathfrak{X}(J^{1*}E)$ is a Cartan-Noether symmetry then

$$\mathrm{L}(X_\mathcal{H})\xi_Y = \mathrm{d}\, i(X_\mathcal{H})\xi_Y - (-1)^m\, i(X_\mathcal{H})\mathrm{d}\xi_Y = -(-1)^m\, i(X_\mathcal{H})\, i(Y)\Omega_h^\nabla = -\, i(Y)\, i(X_\mathcal{H})\Omega_h^\nabla = 0$$

∎

It is interesting to remark that, to our knowledge, given a first integral of a Hamiltonian system, there is no a straightforward way of associating to it a Cartan-Noether symmetry Y. The main obstruction is that, given a $(m-1)$-form ξ, the existence of a solution for the equation $i(Y)\Omega_h^\nabla = \mathrm{d}\xi$ is not assured (even in the case Ω_h^∇ being 1-nondegenerate). Hence, in general, the *converse Noether theorem* cannot be stated for multisymplectic Hamiltonian systems.

Noether's theorem associates first integrals to Cartan-Noether symmetries. But these kinds of symmetries do not exhaust the set of (general) symmetries. As is known, in mechanics there are dynamical symmetries which are not of Cartan type, which generate also conserved quantities (see [29], [32], [33], for some examples). These are the so-called *hidden symmetries*. Different attempts have been made to extend Noether's theorem in order to include these symmetries and the corresponding conserved quantities. Next we present a generalization of the Noether theorem 5, which is based in the approach of reference [35] for mechanical systems.

First we introduce the *higher-order Cartan-Noether symmetries*, generalizing the definition 4 in the following way:

Definition 5 *An* (infinitesimal) Cartan-Noether symmetry of order n *of a Hamiltonian system* $(J^{1*}E, \Omega_h^\nabla)$ *is a vector field* $Y \in \mathfrak{X}(J^{1*}E)$ *satisfying that:*

1. Y *is a general symmetry.*
2. $\mathrm{L}^n(Y)\Omega_h^\nabla = 0$*, but* $\mathrm{L}^k(Y)\Omega_h^\nabla \neq 0$*, for* $k < n$*.*

Observe that Cartan-Noether symmetries of order $n > 1$ are not necessarily Hamiltonian vector

Proposition 7 *If $Y \in \mathfrak{X}(J^{1*}E)$ is a Cartan-Noether symmetry of order n of a Hamiltonian system $(J^{1*}E, \Omega_h^{\nabla})$, then the form $\mathrm{L}^{n-1}(Y)\, i(Y)\Omega_h^{\nabla} \in \Omega^m(J^{1*}E)$ is closed.*

(*Proof*) In fact, from the definition 5 we obtain

$$0 = \mathrm{L}^n(Y)\Omega_h^{\nabla} = \mathrm{L}^{n-1}(Y)\,\mathrm{L}(Y)\Omega_h^{\nabla} = \mathrm{L}^{n-1}(Y)\mathrm{d}\, i(Y)\Omega_h^{\nabla} = \mathrm{d}\,\mathrm{L}^{n-1}(Y)\, i(Y)\Omega_h^{\nabla}$$

■

Hence, this condition is equivalent to demanding that, for every $p \in J^{1*}E$, there exists an open neighborhood $U_p \ni p$, and $\xi_Y \in \Omega^{m-1}(U_p)$, such that $\mathrm{L}^{n-1}(Y)\, i(Y)\Omega_h^{\nabla} = \mathrm{d}\xi_Y$ (on U_p). Then, the result stated in Proposition 5 can be generalized as follows:

Proposition 8 *Let $Y \in \mathfrak{X}(J^{1*}E)$ be a Cartan-Noether symmetry of order n of a Hamiltonian system $(J^{1*}E, \Omega_h^{\nabla})$. Therefore:*

1. *$\mathrm{L}^n(Y)\Theta_h^{\nabla}$ is a closed form, hence, in an open set $U \subset J^{1*}E$, there exist $\zeta_Y \in \Omega^{m-1}(U)$ such that $\mathrm{L}^n(Y)\Theta_h^{\nabla} = \mathrm{d}\zeta_Y$.*
2. *If $\mathrm{L}^{n-1}(Y)\, i(Y)\Omega_h^{\nabla} = \mathrm{d}\xi_Y$, in an open set $U \subset J^{1*}E$, then*
$$\mathrm{L}^n(Y)\Theta_h^{\nabla} = \mathrm{d}(\mathrm{L}^{n-1}(Y)\, i(Y)\Theta_h^{\nabla} - \xi_Y) = \mathrm{d}\zeta_Y \quad (\text{in } U)$$

(*Proof*)

1. The first item is immediate since $\mathrm{d}\,\mathrm{L}^n(Y)\Theta_h^{\nabla} = \mathrm{L}^n(Y)\mathrm{d}\Theta_h^{\nabla} = 0$.
2. For the second item we have
$$\begin{aligned} \mathrm{L}^n(Y)\Theta_h^{\nabla} &= \mathrm{L}^{n-1}(Y)\,\mathrm{L}(Y)\Theta_h^{\nabla} = \mathrm{L}^{n-1}(Y)(\mathrm{d}\, i(Y)\Theta_h^{\nabla} + i(Y)\mathrm{d}\Theta_h^{\nabla}) \\ &= \mathrm{d}\,\mathrm{L}^{n-1}(Y)\, i(Y)\Theta_h^{\nabla} + \mathrm{L}^{n-1}(Y)\, i(Y)\mathrm{d}\Theta_h^{\nabla} \\ &= \mathrm{d}\,\mathrm{L}^{n-1}(Y)\, i(Y)\Theta_h^{\nabla} - \mathrm{d}\xi_Y = \mathrm{d}(\mathrm{L}^{n-1}(Y)\, i(Y)\Theta_h^{\nabla} - \xi_Y) \end{aligned}$$
Hence we can write $\xi_Y = \mathrm{L}^{n-1}(Y)\, i(Y)\Theta_h^{\nabla} - \zeta_Y$.

■

Then, theorem 5 can be generalized for including higher-order Cartan-Noether symmetries:

Theorem 6 (Noether): *If $Y \in \mathfrak{X}(J^{1*}E)$ is a Cartan-Noether symmetry of order n of a Hamiltonian system $(J^{1*}E, \Omega_h^{\nabla})$, with $\mathrm{L}^{n-1}(Y)\, i(Y)\Omega_h^{\nabla} = \mathrm{d}\xi_Y$. Then, for every HDW-multivector field $X_{\mathcal{H}} \in \mathfrak{X}^m(J^{1*}E)$, we have that*

$$\mathrm{L}(X_{\mathcal{H}})\xi_Y = 0$$

that is, the $(m-1)$-form ξ_Y associated with Y is a first integral of $(J^{1*}E, \Omega_h^{\nabla})$.

(*Proof*) If $Y \in \mathfrak{X}(J^{1*}E)$ is a Cartan-Noether symmetry then it is a general symmetry, and then $[Y, X_{\mathcal{H}}] = \mathcal{Z} \in \ker\, \Omega_h^{\nabla}$. Therefore

$$\begin{aligned} \mathrm{L}(X_{\mathcal{H}})\xi_Y &= (-1)^{m+1}\, i(X_{\mathcal{H}})\mathrm{d}\xi_Y = (-1)^{m+1}\, i(X_{\mathcal{H}})\,\mathrm{L}^{n-1}(Y)\, i(Y)\Omega_h^{\nabla} \\ &= (-1)^{m+1}\, i(X_{\mathcal{H}})\,\mathrm{L}(Y)\,\mathrm{L}^{n-2}(Y)\, i(Y)\Omega_h^{\nabla} \\ &= \mathrm{L}(Y)\, i(X_{\mathcal{H}})\,\mathrm{L}^{n-2}(Y)\, i(Y)\Omega_h^{\nabla} - i([Y, X_{\mathcal{H}}])\,\mathrm{L}^{n-2}(Y)\, i(Y)\Omega_h^{\nabla} \end{aligned}$$

and repeating the reasoning $n-2$ times we will arrive at the result

$$\mathrm{L}(X_{\mathcal{H}})\xi_Y = (\mathrm{L}(Y)\,i(X_{\mathcal{H}}) - i(\mathcal{Z}))^{n-1}\,i(Y)\Omega_h^{\nabla} = 0$$

since $i(X_{\mathcal{H}})\,i(Y)\Omega_h^{\nabla} = 0$ and $i(\mathcal{Z})\,i(Y)\Omega_h^{\nabla} = 0$. ■

The study of symmetries of Hamiltonian multisymplectic systems, is, of course, a topic of great interest. The general problem of a group of symmetries acting on a multisymplectic manifold and the subsequent theory of reduction has been analyzed in [17] and [18].

6 Restricted Hamiltonian systems

There are many interesting cases in Field theories where the Hamiltonian field equations are established not in the whole multimomentum phase space $J^{1*}E$, but rather in a submanifold $\mathrm{j}_0\colon P \hookrightarrow J^{1*}E$, such that P is a fiber bundle over E (and M), and the corresponding projections $\tau_0^1\colon P \to E$ and $\bar{\tau}_0^1\colon P \to M$ satisfy that $\tau^1 \circ \mathrm{j}_0 = \tau_0^1$ and $\bar{\tau}^1 \circ \mathrm{j}_0 = \bar{\tau}_0^1$ In that case we will say that $(J^{1*}E, P, \Omega_h^0)$ is a *restricted Hamiltonian system*, where $\Omega_h^0 := \mathrm{j}_0^*\Omega_h^{\nabla}$.

Now we can pose a variational principle in the same way as for the Hamiltonian system $(J^{1*}E, \Omega_h^{\nabla})$, (but with P instead of $J^{1*}E$): the states of the field are the sections of $\bar{\tau}_0^1$ which are critical for the functional $\mathbf{H}_0\colon \Gamma_c(M,P) \to \mathbb{R}$ defined by $\mathbf{H}_0(\psi_0) := \int_M \psi_0^*\Theta_h^0$, for every $\psi_0 \in \Gamma_c(M,P)$. These critical sections will be characterized by the condition (analogous to (5))

$$\psi_0^*\,i(X_0)\Omega_h^0 = 0 \quad , \quad \text{for every } X_0 \in \mathfrak{X}(P)$$

Hence, considering multivector fields, connections and jet fields in P instead of $J^{1*}E$, we have:

Proposition 9 *Let $(J^{1*}E, P, \Omega_h^0)$ be a restricted Hamiltonian system. The critical section of the above variational principle are sections $\psi_0 \in \Gamma_c(M,P)$ satisfying the following equivalent conditions:*

1. *They are the integral sections of an integrable jet field $\mathcal{Y}_{\mathcal{H}}^0\colon P \to J^1P$ satisfying that $i(\mathcal{Y}_{\mathcal{H}}^0)\Omega_h^0 = 0$.*
2. *They are the integral sections of an integrable connection $\nabla_{\mathcal{H}}^0$ satisfying that $i(\nabla_{\mathcal{H}}^0)\Omega_h^0 = (m-1)\Omega_h^0$.*
3. *They are the integral sections of a class of integrable and $\bar{\tau}_0^1$-transverse multivector fields $\{X_{\mathcal{H}}^0\} \subset \mathfrak{X}^m(P)$ such that $i(X_{\mathcal{H}}^0)\Omega_h^0 = 0$, for every $X_{\mathcal{H}}^0 \in \{X_{\mathcal{H}}^0\}$.*

(*Proof*) The proof is like in Theorem 2. ■

Note that the form Ω_h^0 is m-degenerate but, in general, a $\bar{\tau}_0^1$-transverse and locally decomposable multivector field $X_{\mathcal{H}}^0 \in \mathfrak{X}^m(P)$ such that $i(X_{\mathcal{H}}^0)\Omega_h^0 = 0$, does not necessarily exist. Furthermore, the existence of multivector fields of this kind does not imply their integrability. Nevertheless, it is possible for these integrable multivector fields to exist on a submanifold of P. So we can state the following problem: to look for a submanifold $S \hookrightarrow P$ where integrable HDW-multivector fields $X_{\mathcal{H}}^0 \in \mathfrak{X}^m(P)$ exist; and then their integral sections are contained in S.

As a first step, we do not consider the integrability condition. The procedure is algorithmic (from now on we suppose that all the multivector fields are locally decomposable):

- First, let S_1 be the set of points of P where HDW-multivector fields do exist

$$S_1 := \{\tilde{y} \in P \ ; \ \exists X^0_{\mathcal{H}} \in \mathfrak{X}^m(P) \text{ such that} \left\{ \begin{array}{c} (i(X^0_{\mathcal{H}})\Omega^0_h)(\tilde{y}) = 0 \\ (i(X^0_{\mathcal{H}})(\bar{\tau}^{1*}_0\omega))(\tilde{y}) = 1 \end{array} \right\}\}$$

  We assume that S_1 is a non-empty (closed) submanifold of P.

  This is the *compatibility condition*.

- Now, denote by $\mathfrak{X}^m_{HDW}(P, S_1)$ the set of multivector fields in P which are HDW-multivector fields on S_1. Let $X^0_{\mathcal{H}} \colon S_1 \to \Lambda^m \mathrm{T}P|_{S_1}$ be in $\mathfrak{X}^m_{\mathcal{H}}(P, S_1)$. If, in addition, $X^0_{\mathcal{H}} \colon S_1 \to \Lambda^m \mathrm{T}S_1$; that is, $X^0_{\mathcal{H}} \in \mathfrak{X}^m(S_1)$, then we say that X^0 is a solution on S_1. Nevertheless, this last condition is not assured except perhaps in a set of points $S_2 \subset S_1 \subset P$, which we will assume to be a (closed) submanifold, and which is defined by

$$S_2 := \{\tilde{y} \in S_1 \ ; \ \exists X^0_{\mathcal{H}} \in \mathfrak{X}^m_{HDW}(P, S_1) \text{ such that } X^0_{\mathcal{H}}(\tilde{y}) \in \Lambda^m \mathrm{T}_{\tilde{y}} S_1\}$$

  This is the so-called *consistency* or *tangency condition*.

- In this way, a sequence of (closed) submanifolds, $\ldots \subset S_i \subset \ldots \subset S_1 \subset P$, is assumed to be obtained, each one of them being defined as

$$S_i := \{\tilde{y} \in S_{i-1} \ ; \ \exists X^0_{\mathcal{H}} \in \mathfrak{X}^m_{HDW}(P, S_{i-1}) \text{ such that } X^0_{\mathcal{H}}(\tilde{y}) \in \Lambda^m \mathrm{T}_{\tilde{y}} S_{i-1}\}$$

- There are two possible options for the final step of this algorithm, namely:

  1. The algorithm ends by giving a submanifold $S_f \hookrightarrow P$, with $\dim S_f \geq m$, (where $S_f = \bigcap\limits_{i \geq 1} S_i$) and HDW-multivector fields $X^0_{\mathcal{H}} \in \mathfrak{X}^m(S_f)$. S_f is then called the *final constraint submanifold*.
  2. The algorithm ends by giving a submanifold S_f with $\dim S_f < m$, or the empty set. Then there is no HDW-multivector fields $X^0_{\mathcal{H}} \in \mathfrak{X}^m(S_f)$.

This procedure is called the *constraint algorithm*.

The local treatment of this case shows significative differences to the general one. We again have the system of equations for the coefficients $G^{\mu}_{A\nu}$. As we have stated, this system is not compatible in general, and S_1 is the closed submanifold where it is compatible. Then, there are HDW-multivector fields on S_1, but the number of arbitrary functions on which they depend is not the same as in the general case, since it depends on the dimension of S_1. Now the tangency condition must be analyzed in the usual way. Finally, the question of integrability must be considered. To this purpose similar considerations as above must be made for the submanifold S_f instead of $J^{1*}E$.

Some of the problems considered in this and the above section have been treated in an equivalent way, but using Ehresmann connections, in [25] and [26].

As a final remark, concerning to the study of symmetries for restricted Hamiltonian systems, results like those discussed in sections 4 and 5 would be applicable,in general, to this situation, but for the subbundle $S_f \to M$, and taking as symmetries vector fields $Y \in \mathfrak{X}(J^{1*}E$ which are tangent to S_f.

7 Hamiltonian formalism for Lagrangian systems

From the Lagrangian point of view, a Classical Field theory is described by its *configuration bundle*

is usually written as $\mathcal{L} = \pounds\bar{\pi}^{1*}\omega$, where $\pounds \in \mathrm{C}^\infty(J^1E)$ is the *Lagrangian function* associated with $\mathcal{L}$ and ω. Then a *Lagrangian system* is a couple $((E, M; \pi), \mathcal{L})$. The *Poincaré-Cartan m and* $(m+1)$-*forms* associated with the Lagrangian density $\mathcal{L}$ are defined using the *vertical endomorphism* $\mathcal{V}$ of the bundle J^1E [8], [12]:

$$\Theta_{\mathcal{L}} := i(\mathcal{V})\mathcal{L} + \mathcal{L} \equiv \theta_{\mathcal{L}} + \mathcal{L} \in \varOmega^m(J^1E) \quad ; \quad \Omega_{\mathcal{L}} := -\mathrm{d}\Theta_{\mathcal{L}} \in \varOmega^{m+1}(J^1E)$$

In a natural chart in J^1E we have

$$\begin{aligned}
\Theta_{\mathcal{L}} &= \frac{\partial\pounds}{\partial v^A_\mu}\mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu - \left(\frac{\partial\pounds}{\partial v^A_\mu}v^A_\mu - \pounds\right)\mathrm{d}^m x \\
\Omega_{\mathcal{L}} &= -\frac{\partial^2\pounds}{\partial v^B_\nu \partial v^A_\mu}\mathrm{d}v^B_\nu \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu - \frac{\partial^2\pounds}{\partial y^B \partial v^A_\mu}\mathrm{d}y^B \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu + \\
&\quad \frac{\partial^2\pounds}{\partial v^B_\nu \partial v^A_\mu}v^A_\mu \mathrm{d}v^B_\nu \wedge \mathrm{d}^m x + \left(\frac{\partial^2\pounds}{\partial y^B \partial v^A_\mu}v^A_\mu - \frac{\partial\pounds}{\partial y^B} + \frac{\partial^2\pounds}{\partial x^\mu \partial v^B_\mu}\right)\mathrm{d}y^B \wedge \mathrm{d}^m x
\end{aligned}$$

The Lagrangian system is *regular* if $\Omega_{\mathcal{L}}$ is 1-nondegenerate and, as a consequence, $(J^1E, \Omega_{\mathcal{L}})$ is a multisymplectic manifold [4]. Elsewhere the system is *non-regular* or *singular*. The regularity condition is equivalent to demanding that $det\left(\frac{\partial^2\pounds}{\partial v^A_\mu \partial v^B_\nu}(\bar{y})\right) \neq 0$, for every $\bar{y} \in J^1E$. (For more details see, for instance, [2], [5], [8], [12], [13], [14], [34], [36]).

As for Hamiltonian systems, a variational problem can be posed for a Lagrangian system, which is called the *Hamilton principle* of the Lagrangian formalism: the states of the field are the (compact-supported) sections of π which are critical for the functional $\mathbf{L}\colon \Gamma_c(M, E) \to \mathbb{R}$ defined by $\mathbf{L}(\phi) := \int_M (j^1\phi)^*\mathcal{L}$, for every $\phi \in \Gamma_c(M, E)$. These (compact-supported) critical sections are characterized by the condition

$$(j^1\phi)^*\, i(X)\Omega_{\mathcal{L}} = 0 \quad , \quad \text{for every } X \in \mathfrak{X}(J^1E)$$

which, in a natural system of coordinates in J^1E, is equivalent to demanding that ϕ satisfy the *Euler-Lagrange equations*: $\left.\frac{\partial\pounds}{\partial y^A}\right|_{j^1\phi} - \frac{\partial}{\partial x^\mu}\left.\frac{\partial\pounds}{\partial v^A_\mu}\right|_{j^1\phi} = 0$. Then [8], [9], [25], [36]:

Theorem 7 *The critical sections of the Hamilton principle are canonical liftings, $j^1\phi\colon M \to J^1E$, of sections $\phi\colon M \to E$, which satisfy any one of the following conditions:*

1. *They are the integral sections of an holonomic jet field $\mathcal{Y}_{\mathcal{L}}\colon J^1E \to J^1J^1E$ such that $i(\mathcal{Y}_{\mathcal{L}})\Omega_{\mathcal{L}} = 0$.*
2. *They are the integral sections of an holonomic connection $\nabla_{\mathcal{L}}$ such that $i(\nabla_{\mathcal{L}})\Omega_{\mathcal{L}} = (m-1)\Omega_{\mathcal{L}}$.*
3. *They are the integral sections of a class of holonomic multivector fields $\{X_{\mathcal{L}}\} \subset \mathfrak{X}^m(J^1E)$ such that $i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0$, for every $X_{\mathcal{L}} \in \{X_{\mathcal{L}}\}$.*

$X_{\mathcal{L}} \in \mathfrak{X}^m(J^1E)$ is an *Euler-Lagrange multivector field* for $\mathcal{L}$ if it is semi-holonomic and is a solution of the equation $i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0$. (The same terminology is also used for jet fields and connections). Then, using this theorem, it can be proved that [9], [25]:

- If $((E, M; \pi), \mathcal{L})$ is a regular Lagrangian system, then there exist classes of Euler-Lagrange multivector fields for $\mathcal{L}$. In a local system these multivector fields depend on $N(m^2-1)$ arbitrary functions, and they are not integrable necessarily, except perhaps on a submanifold

- For singular Lagrangian systems, the existence of Euler-Lagrange multivector fields is not assured, except perhaps on some submanifold $S \hookrightarrow J^1E$. Furthermore, locally decomposable and $\bar{\pi}^1$-transverse multivector fields, which are solutions of the field equations, can exist (in general, on some submanifold of J^1E), but none of them are semi-holonomic (at any point of this submanifold). As in the regular case, although Euler-Lagrange multivector fields exist on some submanifold S, their integrability is not assured, except perhaps on another smaller submanifold $I \hookrightarrow S$.

The Lagrangian and Hamiltonian formalisms are related by means of the corresponding *Legendre map* $F\mathcal{L}\colon J^1E \to J^{1*}E$. In order to define it, first we introduce the *extended Legendre map* $\widetilde{\mathcal{FL}}\colon J^1E \to \mathcal{M}\pi$ in the following way [26]:

$$(\widetilde{\mathcal{FL}}\bar{y}))(Z_1,\ldots,Z_m) := (\Theta_{\mathcal{L}})_{\bar{y}}(\bar{Z}_1,\ldots,\bar{Z}_m)$$

where $Z_1,\ldots,Z_m \in \mathrm{T}_{\pi^1(\bar{y})}E$, and $\bar{Z}_1,\ldots,\bar{Z}_m \in \mathrm{T}_{\bar{y}}J^1E$ are such that $\mathrm{T}_{\bar{y}}\pi^1\bar{Z}_\mu = Z_\mu$. ($\widetilde{\mathcal{FL}}$ can also be defined as the "first order vertical Taylor approximation to $\pounds$" [5], [15]). Hence, using the natural projection $\mu\colon \mathcal{M}\pi = \Lambda_1^m\mathrm{T}^*E \to \Lambda_1^m\mathrm{T}^*E/\Lambda_0^m\mathrm{T}^*E = J^{1*}E$, we define $F\mathcal{L} := \mu \circ \widetilde{\mathcal{FL}}$. Its local expression is

$$F\mathcal{L}^*x^\mu = x^\mu \quad , \quad F\mathcal{L}^*y^A = y^A \quad , \quad F\mathcal{L}^*p_A^\mu = \frac{\partial \pounds}{\partial v_\mu^A}$$

Definition 6 *Let $((E,M;\pi),\mathcal{L})$ be a Lagrangian system.*

1. *$((E,M;\pi),\mathcal{L})$ is a* regular *or* non-degenerate *Lagrangian system if $F\mathcal{L}$ is a local diffeomorphism. Elsewhere $((E,M;\pi),\mathcal{L})$ is a* singular *or* degenerate *Lagrangian system (This definition is equivalent to that given above).*

   As a particular case, $((E,M;\pi),\mathcal{L})$ is a hyper-regular *Lagrangian system if $F\mathcal{L}$ is a global diffeomorphism.*

2. *A singular Lagrangian system $((E,M;\pi),\mathcal{L})$ is* almost-regular *if:*
   - *(a) $P := \mathrm{F}\mathcal{L}(J^1E)$ is a closed submanifold of $J^{1*}E$.*
     (We will denote the natural imbedding by $\mathrm{j}_0\colon P \hookrightarrow J^{1*}E$).
   - *(b) $F\mathcal{L}$ is a submersion onto its image.*
   - *(c) For every $\bar{y} \in J^1E$, the fibers $F\mathcal{L}^{-1}(F\mathcal{L}(\bar{y}))$ are connected submanifolds of J^1E.*

It can be proved [5], [26], that if $((E,M;\pi),\mathcal{L})$ is a hyper-regular Lagrangian system, then $\widetilde{\mathcal{FL}}(J^1E)$ is a 1-codimensional imbedded submanifold of $\mathcal{M}\pi$, which is transverse to the projection μ, and is diffeomorphic to $J^{1*}E$. This diffeomorphism is $\mathrm{h} := \widetilde{\mathcal{FL}} \circ F\mathcal{L}^{-1}$ (which is just μ^{-1}, when μ is restricted to $\widetilde{\mathcal{FL}}(J^1E)$), and it is a Hamiltonian section. Thus we can construct the Hamilton-Cartan forms by making $\Theta_h = \mathrm{h}^*\Theta$ and $\Omega_h = \mathrm{h}^*\Omega$. Then the couple $(J^{1*}E,\Omega_h)$ is said to be the *Hamiltonian system* associated with the hyper-regular Lagrangian system $((E,M;\pi),\mathcal{L})$. Locally, this Hamiltonian section is specified by the local Hamiltonian function $H = p_A^\mu F\mathcal{L}^{-1^*}v_\mu^A - F\mathcal{L}^{-1^*}\pounds$, then the local expressions of these Hamilton-Cartan forms are (2), and the (non-covariant) expression of the Hamiltonian equations are (3). Of course $F\mathcal{L}^*\Theta_h = \Theta_{\mathcal{L}}$ and $F\mathcal{L}^*\Omega_h = \Omega_{\mathcal{L}}$.

This construction can also be made as follows: given a connection ∇ in the bundle $\pi\colon E \to M$, let $j_\nabla\colon J^{1*}E \to \mathcal{M}\pi$ be the associated linear section, and $\Theta^\nabla = j_\nabla^*\Theta$. Then we can define a unique Hamiltonian density $\mathcal{H}^\nabla$ in two different but equivalent ways: by making the difference $j_\nabla - \mathrm{h}$, or

constructed using the connection ∇. In any case, the form $\Theta_h = \Theta^\nabla - \mathcal{H}^\nabla$, and hence Ω_h, are the same as above (see [11]).

If $((E, M; \pi), \mathcal{L})$ is an almost-regular Lagrangian system, then a *restricted Hamiltonian system* $(J^{1*}E, P, \Omega_h^0)$ can be associated in a similar way [11], [26].

One expects both the Lagrangian and Hamiltonian formalism to be equivalent. As in mechanics, the standard way of showing this equivalence consists in using the Legendre map. First we can lift sections of π from E to $J^{1*}E$, as follows:

Definition 7 *Let $((E, M; \pi), \mathcal{L})$ be a hyper-regular Lagrangian system, $F\mathcal{L}$ the induced Legendre transformation, $\phi\colon M \to E$ a section of π and $j^1\phi\colon M \to J^1E$ its canonical prolongation to J^1E. The* Lagrangian prolongation *of ϕ to $J^{1*}E$ is the section*

$$j^{1*}\phi := F\mathcal{L} \circ j^1\phi\colon M \to J^{1*}E$$

(If $((E, M; \pi), \mathcal{L})$ is an almost-regular Lagrangian system, the Lagrangian prolongation *of a section $\phi\colon M \to E$ to P is $j_0^{1*}\phi := \mathrm{F}\mathcal{L}_0 \circ j^1\phi\colon M \to P$).*

Theorem 8 (Equivalence theorem for sections) *Let $((E, M; \pi), \mathcal{L})$ and $(J^{1*}E, \Omega_h)$ be the Lagrangian and Hamiltonian descriptions of a hyper-regular system.*

If a section $\phi \in \Gamma_c(M, E)$ is a solution of the Lagrangian variational problem (Hamilton principle) then the section $\psi \equiv j^{1*}\phi := F\mathcal{L} \circ j^1\phi \in \Gamma_c(M, J^{1*}E)$ is a solution of the Hamiltonian variational problem (Hamilton-Jacobi principle).

Conversely, if $\psi \in \Gamma_c(M, J^{1*}E)$ is a solution of the Hamiltonian variational problem, then the section $\phi \equiv \tau^1 \circ \psi \in \Gamma_c(M, E)$ is a solution of the Lagrangian variational problem.

(*Proof*) Bearing in mind the diagram

$$\begin{tikzcd}
J^1E \arrow[rr, "F\mathcal{L}"] \arrow[dr, "\pi^1"] & & J^{1*}E \arrow[dl, "\tau^1"'] \\
& E \arrow[d, shift left, "\pi"] & \\
& M \arrow[u, shift left, "\phi"] \arrow[uul, "j^1\phi"] \arrow[uur, "\psi"'] &
\end{tikzcd} \tag{13}$$

If ϕ is a solution of the Lagrangian variational problem then $(j^1\phi)^*\, i(X)\Omega_\mathcal{L} = 0$, for every $X \in \mathfrak{X}(J^1E)$ (Theorem 7); therefore, as $F\mathcal{L}$ is a local diffeomorphism,

$$\begin{aligned}
0 &= (j^1\phi)^*\, i(X)\Omega_\mathcal{L} = (j^1\phi)^*\, i(X)(F\mathcal{L}^*\Omega_h) \\
&= (j^1\phi)^* F\mathcal{L}^*(i(F\mathcal{L}_*^{-1}X)\Omega_h) = (F\mathcal{L} \circ j^1\phi)^*\, i(X')\Omega_h)
\end{aligned}$$

which holds for every $X' \in \mathfrak{X}(J^{1*}E)$ and thus, by (5), $\psi \equiv F\mathcal{L} \circ j^1\phi$ is a solution of the Hamiltonian variational problem. (This proof also holds for the almost-regular case).

Conversely, let $\psi \in \Gamma_c(M, J^{1*}E)$ be a solution of the Hamiltonian variational problem. Reversing the above reasoning we obtain that $(\mathrm{F}\mathcal{L}^{-1} \circ \psi)^*\, i(X)\Omega_\mathcal{L} = 0$, for every $X \in \mathfrak{X}(J^1E)$, and hence

we are in the hyper-regular case, σ must be an holonomic section, $\sigma = j^1\phi$ [9], [25], [36], and since (13) is a commutative diagram, $\phi = \tau^1 \circ \psi \in \Gamma_c(M,E)$. ■

Observe that every section $\psi: M \to J^{1*}E$ which is solution of the Hamilton-Jacobi variational principle is necessarily a Lagrangian prolongation of a section $\phi: M \to E$.

Theorem 9 *Let $((E,M;\pi),\mathcal{L})$ and $(J^{1*}E,\Omega_h)$ be the Lagrangian and Hamiltonian descriptions of a hyper-regular system.*

1. (Equivalence theorem for jet fields and connections) *Let $\mathcal{Y}_\mathcal{L}$ and $\mathcal{Y}_\mathcal{H}$ be the jet fields solution of the Lagrangian and Hamiltonian field equations respectively. Then*

$$j^1F\mathcal{L} \circ \mathcal{Y}_\mathcal{L} = \mathcal{Y}_\mathcal{H} \circ F\mathcal{L}$$

(we say that the jet fields $\mathcal{Y}_\mathcal{L}$ and $\mathcal{Y}_\mathcal{H}$ are $F\mathcal{L}$-related). As a consequence, their associated connection forms, $\nabla_\mathcal{L}$ and $\nabla_\mathcal{H}$ respectively, are $F\mathcal{L}$-related too.

2. (Equivalence theorem for multivector fields) *Let $X_\mathcal{L} \in \mathfrak{X}^m(J^1E)$ and $X_\mathcal{H} \in \mathfrak{X}^m(J^{1*}E)$ be multivector fields solution of the Lagrangian and Hamiltonian field equations respectively. Then*

$$\Lambda^m \mathrm{T}F\mathcal{L} \circ X_\mathcal{L} = f X_\mathcal{H} \circ F\mathcal{L}$$

for some $f \in \mathrm{C}^\infty(J^{1*}E)$ (we say that the classes $\{X_\mathcal{L}\}$ and $\{X_\mathcal{H}\}$ are $F\mathcal{L}$-related).

That is, we have the following (commutative) diagrams:

$$\begin{tikzcd} \Lambda^m \mathrm{T}J^1E \arrow[r, "\Lambda^m \mathrm{T}F\mathcal{L}"'] & \Lambda^m \mathrm{T}J^{1*}E \\ J^1E \arrow[u, "X_\mathcal{L}"] \arrow[r, "F\mathcal{L}"] & J^{1*}E \arrow[u, "X_\mathcal{H}"'] \end{tikzcd} \quad ; \quad \begin{tikzcd} J^1J^1E \arrow[r, "j^1F\mathcal{L}"'] & J^1(J^{1*}E) \\ J^1E \arrow[u, "\mathcal{Y}_\mathcal{L}"] \arrow[r, "F\mathcal{L}"] & J^{1*}E \arrow[u, "\mathcal{Y}_\mathcal{H}"'] \end{tikzcd}$$

(*Proof*) The first item is a consequence of Theorem 8, since the critical sections solutions of the Lagrangian and Hamiltonian variational problems (which are $F\mathcal{L}$-related) are the integral sections of the jet fields $\mathcal{Y}_\mathcal{L}$ and $\mathcal{Y}_\mathcal{H}$, respectively (see also [26]).

The second item is an immediate consequence of the first one and the equivalence between orientable and integrable jet fields and classes of non-vanishing, locally decomposable, transverse and integrable multivector fields. ■

8 Example

(See also the reference [34]).

Most of the (quadratic) Lagrangian systems in field theories can be modeled as follows: $\pi: E \to M$ is a trivial bundle (usually $E = M \times \mathbb{R}^N$) and then $\pi^1: J^1E \to E$ is a vector bundle. g is a metric in this vector bundle, γ is a connection of the projection π^1, and $f \in \mathrm{C}^\infty(E)$ is a potential function. Then the Lagrangian function is

$$\ell(\bar{y}) = \frac{1}{2}(\bar{y} - \gamma(\pi^1(\bar{y})), \bar{y} - \gamma(\pi^1(\bar{y}))) - (\pi^{1*}f)(\bar{y}) \quad (\text{for } \bar{y} \in J^1E)$$

and in natural coordinates takes the form [34]

$$\pounds = \frac{1}{2} a^{\mu\nu}_{AB}(y)(v^A_\mu - \gamma^A_\mu(x))(v^B_\nu - \gamma^B_\nu(x)) + f(y)$$

For simplicity, we consider a model where the matrix of the coefficients $a^{\mu\nu}_{AB}$ is regular and symmetric at every point (that is, $a^{\mu\nu}_{AB}(y) = a^{\nu\mu}_{BA}(y)$). This fact is equivalent to the non-degeneracy of the metric g. The Legendre map associated with this Lagrangian system is given by

$$F\mathcal{L}^* x^\mu = x^\mu \quad , \quad F\mathcal{L}^* y^A = y^A \quad , \quad F\mathcal{L}^* p^\mu_A = a^{\mu\nu}_{AB}(y)(v^B_\nu - \gamma^B_\nu(x))$$

and the local expression of the Hamilton-Cartan $(m+1)$-form is (2), where the local Hamiltonian function is

$$H = \frac{1}{2}\tilde{a}^{AB}_{\mu\nu}(y) p^\mu_A p^\nu_B - f(y)$$

(here $\tilde{a}^{AB}_{\mu\nu}$ denote the coefficients of the inverse matrix of $(a^{\mu\nu}_{AB})$). Hence

$$\begin{aligned} \Theta^\nabla_h &= p^\mu_A \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu - \left(\frac{1}{2}\tilde{a}^{AB}_{\mu\nu}(y) p^\mu_A p^\nu_B - f(y)\right) \mathrm{d}^m x \\ \Omega^\nabla_h &= -\mathrm{d}p^\mu_A \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu + \mathrm{d}\left(\frac{1}{2}\tilde{a}^{AB}_{\mu\nu}(y) p^\mu_A p^\nu_B - f(y)\right) \wedge \mathrm{d}^m x \end{aligned}$$

and it is a multisymplectic form. Then, taking (7) as the local expression for representatives of the corresponding classes of HDW-multivector fields $\{X_\mathcal{H}\} \subset \mathfrak{X}^m_{HDW}(J^{1*}E)$, their components F^A_μ are

$$F^A_\mu = \frac{\partial H}{\partial p^\mu_A} = \tilde{a}^{AB}_{\mu\nu}(y) p^\nu_B$$

and $G^\mu_{A\nu}$ are related by the equations

$$G^\rho_{A\rho} = -\frac{\partial H}{\partial y^A} = -\frac{1}{2}\frac{\partial \tilde{a}^{CB}_{\mu\nu}}{\partial y^A} p^\mu_C p^\nu_B + \frac{\partial f}{\partial y^A} \tag{14}$$

This system allows us to isolate N of these components as functions of the remaining $N(m^2-1)$; and then it determines a family of (classes of) HDW-multivector fields. In order to obtain an integrable class, the condition of integrability $\mathcal{R} = 0$ (where $\mathcal{R}$ is the curvature of the associated connection) must hold; that is, equations (10) and (11) must be added to the last system.

As a simpler case, we consider that the matrix of coefficients is $\tilde{a}^{AB}_{\mu\nu}(y) = \delta^{AB}\delta_{\mu\nu}$, (that is, we take an orthonormal frame for the metric g), then we have that $H = \frac{1}{2}\delta^{AB}\delta_{\mu\nu}p^\mu_A p^\nu_B - f(y)$. In this case, equations (14) reduce to

$$G^\rho_{A\rho} = \frac{\partial f}{\partial y^A}$$

From this system we can isolate N of the coefficients $G^\mu_{A\nu}$; for instance, if $\mu, \nu = 0, 1, \ldots, m-1$, those for which $\mu = \nu = 0$: Thus

$$G^0_{A0} = \frac{\partial f}{\partial y^A} - \sum_{\mu=1}^{m-1} \delta^{AB} G^\mu_{B\mu}$$

Therefore the HDW-multivector fields are

$$\begin{aligned} X_\mathcal{H} &= \bigwedge_{\mu=0}^{m-1}\left(\frac{\partial}{\partial x^\mu} + \delta^{AB}\delta_{\mu\nu}p^\nu_B \frac{\partial}{\partial y^A} + \delta^0_\mu \left[\frac{\partial f}{\partial y^A} - \sum_{\nu=1}^{m-1}\delta^{AB}G^\mu_{B\mu}\right]\frac{\partial}{\partial p^0_A}\right. \\ &\quad \left. + \sum G^\mu_{B\eta}\frac{\partial}{\partial p^\eta} + \sum G^\mu_{C\eta}\frac{\partial}{\partial p^\eta}\right) \end{aligned}$$

Now, if we look for integrable Euler-Lagrange multivector fields, the integrability conditions (10) and (11) must be imposed.

The Lagrangian formalism for this model (using multivector fields) has been studied in [9]. Then, the corresponding (semi-holonomic) Euler-Lagrange multivector fields $X_{\mathcal{L}}$ given there by

$$\begin{aligned} X_{\mathcal{L}} \ &= \ \bigwedge_{\mu=0}^{m-1}\left(\frac{\partial}{\partial x^\mu} + v^A_\mu \frac{\partial}{\partial y^A} + \delta_{0\mu}\delta^{AC}\left[\frac{\partial f}{\partial y^C} - \sum_{\nu=1}^{m-1}\delta_{CD}\bar{G}^D_{\nu\nu}\right]\frac{\partial}{\partial v^A_0}\right. \\ &\left. \quad + \sum_{\mu=\eta\neq 0}\bar{G}^B_{\mu\eta}\frac{\partial}{\partial v^B_\eta} + \sum_{\mu\neq\eta}\bar{G}^C_{\mu\eta}\frac{\partial}{\partial v^C_\eta}\right) \end{aligned}$$

can be compared with the HDW-multivector fields here obtained, observing that, in fact, they are related as stated in the second item of Theorem 9.

As a final remark, we can obtain some typical first integrals, by applying Noether's theorem. As infinitesimal generators of symmetries we take the following π-projectable vector fields in E

$$Z_\mu = \frac{\partial}{\partial x^\mu} \quad , \quad Z_{\mu\nu} = x^\mu \frac{\partial}{\partial x^\nu} - x^\nu \frac{\partial}{\partial x^\mu}$$

(they are isometries of the metric g and symmetries of the potential function f, which generate space-time translations and rotations), and whose canonical liftings to $J^{1*}E$ are the vector fields

$$Y_\mu = \frac{\partial}{\partial x^\mu} \quad , \quad Y_{\mu\nu} = x^\mu \frac{\partial}{\partial x^\nu} - x^\nu \frac{\partial}{\partial x^\mu} + p^\nu_A \frac{\partial}{\partial p^\mu_A} - p^\mu_A \frac{\partial}{\partial p^\nu_A}$$

In fact, they are Cartan-Noether symmetries satisfying that $\mathrm{L}(Y_\mu)\Theta^\nabla_h = 0$ and $\mathrm{L}(Y_{\mu\nu})\Theta^\nabla_h = 0$, and their corresponding associated first integrals are then

$$\begin{aligned} \xi_{Y_\mu} \ &= \ i(Y_\mu)\Theta^\nabla_h = -p^\rho_A \mathrm{d}y^A \wedge \mathrm{d}^{m-2}x_{\mu\rho} + H\mathrm{d}^{m-1}x_\mu \\ \xi_{Y_{\mu\nu}} \ &= \ i(Y_{\mu\nu})\Theta^\nabla_h = x^\mu(-p^\rho_A \mathrm{d}y^A \wedge \mathrm{d}^{m-2}x_{\nu\rho} + H\mathrm{d}^{m-1}x_\nu) - x^\nu(-p^\rho_A \mathrm{d}y^A \wedge \mathrm{d}^{m-2}x_{\mu\rho} + H\mathrm{d}^{m-1}x_\mu) \\ &= \ x^\mu \xi_{Y_\nu} - x^\nu \xi_{Y_\mu} \end{aligned}$$

If $S \hookrightarrow J^{1*}E$ is an integral submanifold of the system, this means that

$$j^*_S \mathrm{d}\xi_{Y_\mu} = 0 \quad , \quad j^*_S \mathrm{d}(x^\mu \xi_{Y_\nu} - x^\nu \xi_{Y_\mu}) = \mathrm{d}x^\mu \wedge j^*_S \xi_{Y_\nu} - \mathrm{d}x^\nu \wedge j^*_S \xi_{Y_\mu} = 0$$

9 Conclusions

We have used the relation between jet fields (connections) and multivector fields in jet bundles to give alternative geometric formulations of the Hamiltonian equations of first-order Classical Field theories, and study their characteristic features. In particular:

- The difference between the Hamilton-De Donder-Weyl equations and the covariant form of the Hamiltonian equations is analyzed and throughly clarified from a geometrical point of view.
- We prove that the Hamiltonian field equations can be written in three equivalent geometric ways: using multivector fields in $J^{1*}E$ (the multimomentum bundle of the Hamiltonian formalism), jet fields in $J^1(J^{1*}E)$ or their associated Ehresmann connections in $J^{1*}E$. These descriptions allow us to write these field equations in an analogous way to the dynamical

- Using the formalism with multivector fields, we show that the field equations $i(X_{\mathcal{H}})\Omega_h^{\nabla} = 0$, with $X_{\mathcal{H}} \in \mathfrak{X}^m(J^{1*}E)$ locally decomposable and $\bar{\tau}^1$-transverse, have solution everywhere in $J^{1*}E$, which is not unique; that is, there are classes of *HDW multivector fields* which are solution of these equations. Nevertheless, these multivector fields are not necessarily integrable everywhere in $J^{1*}E$. These features are significant differences in relation to the analogous situation in mechanics.

- The concept of (infinitesimal) symmetry of a Hamiltonian system $(J^{1*}E, \Omega_h^{\nabla})$ in Field theory is introduced and discussed from different points of view. The relation between *Cartan-Noether symmetries* (those leading to first integrals of Noether type) and *general symmetries* has been discussed.

- In particular, a version of Noether's theorem (in the Hamiltonian formalism) using multivector fields is proved. This statement is also generalized in order to include first integrals arising from higher-order Cartan-Noether symmetries.

- We have analyzed the case of *restricted Hamiltonian systems* (i.e., those such that the Hamiltonian equations are stated in a subbundle $P \to E \to M$ of $J^{1*}E$). In this case, not even the existence of HDW-multivector field is assured, and an algorithmic procedure in order to obtain a submanifold of P where HDW-multivector fields exist, is outlined. Of course the solution is not unique, in general.

- For Hamiltonian systems associated with hyper-regular Lagrangian systems in Field theory, we have proved different versions of the one-to-one correspondence between the solutions of field equations in both formalisms; namely: the *equivalence theorem* for sections, jet fields and connections, and multivector fields.

Hence, this work completes the results of [9], where the special features of the Lagrangian formalism of first-order Field theories in terms of multivector fields were studied.

A Appendix

(See [9], and also [3], [4] and [19]).

Let E be a n-dimensional differentiable manifold. Sections of $\Lambda^m(\mathrm{T}E)$ (with $1 \leq m \leq n$) are called *m-multivector fields* in E. We will denote by $\mathfrak{X}^m(E)$ the set of m-multivector fields in E. Given $Y \in \mathfrak{X}^m(E)$, for every $p \in E$, there exists an open neighborhood $U_p \subset E$ and $Y_1, \ldots, Y_r \in \mathfrak{X}(U_p)$ such that

$$Y \underset{U_p}{=} \sum_{1 \leq i_1 < \ldots < i_m \leq r} f^{i_1 \ldots i_m} Y_{i_1} \wedge \ldots \wedge Y_{i_m}$$

with $f^{i_1 \ldots i_m} \in \mathrm{C}^\infty(U_p)$ and $m \leq r \leq \dim E$. A multivector field $Y \in \mathfrak{X}^m(E)$ is *locally decomposable* if, for every $p \in E$, there exists an open neighborhood $U_p \subset E$ and $Y_1, \ldots, Y_m \in \mathfrak{X}(U_p)$ such that $Y \underset{U_p}{=} Y_1 \wedge \ldots \wedge Y_m$.

If $\Omega \in \Omega^k(E)$ is a differentiable k-form in E, we can define the contraction

$$i(Y)\Omega \underset{U_p}{=} \sum_{1 \leq i_1 < \ldots < i_m \leq r} f^{i_1 \ldots i_m} i(Y_1 \wedge \ldots \wedge Y_m)\Omega = \sum_{1 \leq i_1 < \ldots < i_m \leq r} f^{i_1 \ldots i_m} i(Y_1) \ldots i(Y_m)\Omega$$

if $k \geq m$, and equal to zero if $k < m$. The k-form Ω is said to be *j-nondegenerate* (for $1 \leq j \leq k-1$) if, for every $p \in E$ and $Y \in \mathfrak{X}^j(E)$, $i(Y_p)\Omega_p = 0 \Leftrightarrow Y_p = 0$. The graded bracket

defines an operation of degree $m-1$ which is called the *Lie derivative* respect to Y. If $Y \in \mathfrak{X}^i(E)$ and $X \in \mathfrak{X}^j(E)$, the graded commutator of $\mathrm{L}(Y)$ and $\mathrm{L}(X)$ is another operation of degree $i+j-2$ of the same type, i.e., there will exists a $(i+j-1)$-multivector denoted by $[Y, X]$ such that,

$$[\mathrm{L}(Y), \mathrm{L}(X)] = \mathrm{L}([Y, X])$$

The bilinear assignment $X, Y \mapsto [X, Y]$ is called the *Schouten-Nijenhuis bracket* of X, Y. If X, Y and Z are multivector fields of degrees i, j, k, respectively, then the following properties hold:

1. $[X, Y] = -(-1)^{(i+1)(j+1)}[Y, X]$.
2. $[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{(i+1)(j+1)} Y \wedge [X, Z]$.
3. $(-1)^{(i+1)(k+1)}[X, [Y, Z]] + (-1)^{(j+1)(i+1)}[Y, [Z, X]] + (-1)^{(k+1)(j+1)}[Z, [X, Y]] = 0$.

Moreover, if $X \in \mathfrak{X}^l(E)$ and $Y \in \mathfrak{X}^m(E)$, then

$$i([X, Y])\Omega = \mathrm{L}(X)\, i(Y)\Omega - (-1)^{l+m}\, i(Y)\, \mathrm{L}(X)\Omega$$

A non-vanishing m-multivector field $Y \in \mathfrak{X}^m(E)$ and a m-dimensional distribution $D \subset \mathrm{T}E$ are *locally associated* if there exists a connected open set $U \subseteq E$ such that $Y|_U$ is a section of $\Lambda^m D|_U$. If $Y, Y' \in \mathfrak{X}^m(E)$ are non-vanishing multivector fields locally associated with the same distribution D, on the same connected open set U, then there exists a non-vanishing function $f \in \mathrm{C}^\infty(U)$ such that $Y' \underset{U}{=} fY$. This fact defines an equivalence relation in the set of non-vanishing m-multivector fields in E, whose equivalence classes will be denoted by $\{Y\}_U$. Then, there is a bijective correspondence between the set of m-dimensional orientable distributions D in $\mathrm{T}E$ and the set of the equivalence classes $\{Y\}_E$ of non-vanishing, locally decomposable m-multivector fields in E. The distribution associated with the class $\{Y\}_U$ is denoted $\mathcal{D}_U(Y)$. If $U = E$ we write $\mathcal{D}(Y)$.

A submanifold $S \hookrightarrow E$, with $\dim S = m$, is said to be an integral manifold of $Y \in \mathfrak{X}^m(E)$ if, for every $p \in S$, Y_p spans $\Lambda^m \mathrm{T}_p S$. Y is an integrable multivector field on an open set $U \subseteq E$ if, for every $p \in U$, there exists an integral manifold $S \hookrightarrow U$ of Y, with $p \in S$. Y is integrable if it is integrable in E. Y is involutive on a connected open set $U \subseteq E$ if it is locally decomposable in U and its associated distribution $\mathcal{D}_U(Y)$ is involutive. Y is involutive if it is involutive on E. If a $Y \in \mathfrak{X}^m(E)$ is integrable, then so is every other in its equivalence class $\{Y\}$, and all of them have the same integral manifolds. Moreover, *Frobenius' theorem* allows us to say that a non-vanishing and locally decomposable multivector field is integrable on a connected open set $U \subseteq E$ if, and only if, it is involutive on U.

Now, let $\pi: E \to M$ be a fiber bundle. $Y \in \mathfrak{X}^m(E)$ is said to be π*-transverse* if, at every point $y \in E$, $(i(Y)(\pi^*\omega))_y \neq 0$, for every $\omega \in \Omega^m(M)$ with $\omega(\pi(y)) \neq 0$. Then, if $Y \in \mathfrak{X}^m(E)$ is integrable, Y is π-transverse if, and only if, its integral manifolds are local sections of $\pi: E \to M$. In this case, if $\phi: U \subset M \to E$ is a local section with $\phi(x) = y$ and $\phi(U)$ is the integral manifold of Y through y, then $\mathrm{T}_y(\mathrm{Im}\,\phi)$ is $\mathcal{D}_y(Y)$.

In Hamiltonian Field theory we are interested in multivector fields in $\bar{\tau}^1: J^{1*}E \to M$. Now remember that a *connection* in $J^{1*}E$ is one of the following equivalent elements: a global section $\mathcal{Y}: J^{1*}E \to J^1(J^{1*}E)$ of the projection $J^1(J^{1*}E) \to J^{1*}E$ (a *jet field*), a subbundle $\mathrm{H}(J^{1*}E)$ of $\mathrm{T}J^{1*}E$ such that $\mathrm{T}J^{1*}E = \mathrm{V}(\bar{\tau}^1) \oplus \mathrm{H}(J^{1*}E)$ (which is called a *horizontal subbundle*, and it is also denoted by $\mathcal{D}(\mathcal{Y})$ when considered as the distribution associated with $\mathcal{Y}$), or a $\bar{\tau}^1$-semibasic 1-form ∇ on $J^{1*}E$ with values in $\mathrm{T}J^{1*}E$, such that $\nabla^*\alpha = \alpha$, for every $\bar{\tau}^1$-semibasic form $\alpha \in \Omega^1(J^{1*}E)$ (the *connection form* or *Ehresmann connection*). A jet field $\mathcal{Y}: J^{1*}E \to J^1(J^{1*}E)$ (or a connection

Theorem 10 *There is a bijective correspondence between the set of orientable jet fields* $\mathcal{Y}\colon J^{1*}E \to J^1(J^{1*}E)$ *(or orientable connections* ∇ *in* $\bar{\tau}^1\colon J^{1*}E \to M$*) and the set of the equivalence classes of locally decomposable and* $\bar{\tau}^1$*-transverse multivector fields* $\{X\} \subset \mathfrak{X}^m(J^{1*}E)$ *(they are characterized by the fact that* $\mathcal{D}(\mathcal{Y}) = \mathcal{D}(X)$*). Then,* $\mathcal{Y}$ *is integrable, if, and only if, so is* X*, for every* $X \in \{X\}$*.*

The expression for a representative multivector field X of the class $\{X\}$ associated with a jet field $\mathcal{Y} \equiv (x^\mu, y^A, p^\mu_A, F^A_\mu(x,y,p), G^\rho_{A\mu}(x,y,p))$ is $X = \bigwedge_{\mu=1}^m \left(\frac{\partial}{\partial x^\mu} + F^A_\mu \frac{\partial}{\partial y^A} + G^\rho_{A\mu}\frac{\partial}{\partial p^\rho_A}\right)$.

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